Analytical theory of bioheat transport

Jing Fan and Liqiu Wang[a)]
Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong

(Received 22 February 2011; accepted 19 March 2011; published online 20 May 2011)

Macroscale thermal models for biological tissues can be developed either by the mixture theory of continuum mechanics or by the porous-media theory. Characterized by its simplicity, the former applies scaling-down from the global scale. The latter uses scaling-up from the microscale by the volume averaging, thus offers the connection between microscale and macroscale properties and is capable of describing the rich blood-tissue interaction in biological tissues. By using the porous-media approach, a general bioheat transport model is developed with the required closure provided. Both blood and tissue macroscale temperature fields are shown to satisfy the dual-phase-lagging (DPL) energy equations. Thermal waves and possible resonance may appear due to the coupled conduction between blood and tissue. For the DPL bioheat transport, contributions of the initial temperature distribution, the source term and the initial rate of change of temperature are shown to be inter-expressible under linear boundary conditions. This reveals the solution structure and considerably simplifies the development of solutions of the DPL bioheat equations. Effectiveness and features of the developed solution structure theorems are demonstrated via examining bioheat transport in skin tissue and during magnetic hyperthermia. © *2011 American Institute of Physics*. [doi:10.1063/1.3580330]

I. INTRODUCTION

The study of heat transport in biological tissues has always been and will continually be a significant but difficult problem. It is well recognized that the change of local temperature has considerable effect on the rates of nearly all physiological functions[1,2] so that many therapeutic or diagnostic procedures target temperature as a primary controlling or monitoring parameter, such as hyperthermia,[3] cryosurgery,[4] laser irradiation,[5] and temperature-based disease diagnostics.[6] Unfortunately, accurate quantification of bio-transport processes is rather difficult due to the complex thermal interaction between vascular and extra-vascular systems. The complexity comes from several factors peculiar to living tissues, including the intricate anatomical structure, the blood flow in vessels, and the blood perfusion. More sophistically, these factors are sensitive to outside influence such as temperature.

Some proposed bio-transport models inclusive of all early models regard the tissues of interest as a continuum in which the large number of vessels are collectively accounted to avoid considering the microscopic anatomical structure. The governing equations are for the macroscopic temperature field with several source terms describing the thermal interaction between blood and surrounding tissues (e.g., blood convective heat transfer and blood perfusion) as well as the production and external supply of heat (e.g., metabolic heat generation and external heat supply during therapeutic procedures). This is consistent with our general interests in the phenomenological scale (macroscale) rather than molecular scale or microscale of heat transport for practical applications.

The development of a bioheat model by using the continuum approach is based on the macroscale point equation of energy conservation:

$$\frac{\partial(\rho c T)_{mac}}{\partial t} = \nabla \cdot \mathbf{q}_{mac} + q_{m,mac} + q_{c,mac} + q_{p,mac} + q_{e,mac}, \tag{1.1}$$

where the subscript *mac* is used to indicate the macroscale properties. ρ and c are density and specific heat, respectively. T denotes the temperature and t the time. $\mathbf{q}$ denotes the heat flux density vector. q_m, q_c, and q_p are the volumetric rates of heat generation by the metabolic heating, the blood convective heat transfer and the blood perfusion, respectively. q_e is the volumetric rate of external heat supply like the one used in hyperthermia therapy. Two major issues are thus: (i) the selection of the constitutive relation of heat flux density, and (ii) the expressions of the source terms to describe the interaction between blood and surrounding tissues. Note that the temperature of the extravascular tissues is often identified as the temperature of the continuum based on the argument that blood occupies only a small portion so that it has little effect on the continuum's temperature.

As the first constitutive relation of heat flux density, Fourier's law [Eq. (1.2)] has been used to build macroscale bioheat model since the pioneering work of Pennes.[7]

$$\mathbf{q}(\mathbf{r}, t) = -k\nabla T(\mathbf{r}, t), \tag{1.2}$$

where $\mathbf{r}$ stands for the material point, k is the thermal conductivity, and ∇ is the gradient operator. Some other models were also proposed by assuming Fourier heat conduction, but with different descriptions for the blood-tissue interaction. The Fourier's-law-based bioheat model reads:

$$\frac{\partial(\rho c T)_{mac}}{\partial t} = -\nabla \cdot (k\nabla T)_{mac} + q_{m,mac} + q_{c,mac} + q_{p,mac} + q_{e,mac}. \tag{1.3}$$

Table I summarizes the expressions of the four source terms in Eq. (1.3) in the typical continuum models proposed by

[a)] Electronic mail: lqwang@hku.hk.

TABLE I. Source terms in Pennes, Wulff, Klinger and Chen and Holmes models (superscript b and t indicating blood properties and tissue properties, respectively; ω^b: blood perfusion rate; T^a: temperature of the arterial blood supply; v_h: local mean blood velocity; Δh: enthalpy of formation in metabolic reaction; ϕ: extent of reaction; $\mathbf{v}_p$: blood mean permeation velocity; ω^*: blood perfusion rate only in the small vessels that are collectively treated; T^{a*}: temperature of the arterial blood supply to the small vessels).

Model	$q_{m,mac}$	$q_{c,mac}$	$q_{p,mac}$	$q_{e,mac}$
Pennes	$q_{m,mac}$	0	$(\rho c)^b_{mac}\omega^b(T^a_{mac}-T^t_{mac})$	0
Wulff	$\rho^b_{mac}v_h\Delta h\nabla\phi$	$-(\rho c)^b_{mac}v_h\nabla T^t_{mac}$	0	0
Klinger	$q_{m,mac}$	$-(\rho c)^b_{mac}\mathbf{v}^b_{mac}\nabla T^t_{mac}$	0	0
Chen and Holmes	$q_{m,mac}$	$-(\rho c)^b_{mac}\mathbf{v}_p\cdot\nabla T^t_{mac}$	$(\rho c)^b_{mac}\omega^*(T^a_{mac}{}^*-T^t_{mac})$	0

Pennes,[7] Wulff,[8] Klinger,[9,10] and Chen and Holmes.[11] Pennes[7] neglected the convective heat flux q_c based on the assumption of very low blood velocity and postulated that heat transfer between blood and surrounding tissues mainly occurs in the capillary bed so that the blood perfusion heat flux q_p can be modeled as an isotropic heat source which is proportional to the blood perfusion rate and the temperature difference between the local tissue and the arterial blood supply. Wulff,[8] Klinger,[9,10] and Chen and Holmes[11] questioned Pennes' assumptions, included the convection heat flux in the models and thus considered the effects of blood flow direction on bioheat transfer processes.

With the development of high-intensity and extremely-short-duration heating technologies, the hypothesis of infinite heat propagation speed in the Fourier's law becomes unacceptable. Cattaneo[12] and Vernotte[13,14] proposed the so-called CV constitutive relation:

$$\mathbf{q}(\mathbf{r},t)+\tau_q\frac{\partial\mathbf{q}(\mathbf{r},t)}{\partial t}=-k\nabla T(\mathbf{r},t),\tag{1.4}$$

where $\tau_q>0$ is a material property called the relaxation time. The CV relation is a first-order approximation of the single-phase-lagging model[15]

$$\mathbf{q}(\mathbf{r},t+\tau_q)=-k\nabla T(\mathbf{r},t)\tag{1.5}$$

according to which the temperature gradient established at a point $\mathbf{r}$ at time t gives rise to a heat flux vector at $\mathbf{r}$ at a later time $t+\tau_q$. The corresponding heat conduction equation by applying the CV constitutive relation to Eq. (1.1) is usually called the thermal wave model of bioheat transfer (TWMBT) or the hyperbolic bioheat model, which is of hyperbolic type, characterizing the combined diffusion and wavelike behavior of heat conduction and predicting a finite speed of heat propagation.[16] In the existing TWMBT, the source terms in Eq. (1.1) are usually treated similarly as the Pennes model: neglecting the convection heat flux and including an isotropic blood perfusion source.[17–19] Assuming constant physical properties, the model has the form of

$$\begin{aligned}&\left[1+\tau_q\frac{(\rho c\omega)^b_{mac}}{(\rho c)_{mac}}\right]\frac{\partial T_{mac}}{\partial t}+\tau_q\frac{\partial^2 T_{mac}}{\partial t^2}\\&=\alpha_{mac}\nabla^2T_{mac}+\frac{(\rho c\omega)^b_{mac}}{(\rho c)_{mac}}\left(T^a_{mac}-T_{mac}\right)\\&+\frac{1}{(\rho c)_{mac}}\left(1+\tau_q\frac{\partial}{\partial t}\right)\left(q_{m,mac}+q_{e,mac}\right)\end{aligned}\tag{1.6}$$

where α is the thermal diffusivity. ω^b and T^a, as defined in Table I, are blood perfusion rate and arterial blood temperature, respectively. TWMBT has been applied to analyze different types of bioheat transfer processes, such as the temperature variation in radio frequency heating and pulsed laser treatment,[19] the temperature and thermal dose distributions in living tissues during thermal therapies,[18] the prediction of thermal stresses in skin during cryopreservation,[20] the temperature and thermal damage distributions in skin tissue under different heating conditions.[17,21–23] Based on the reported experimental values of τ_q for biological systems, which can be up to more than 10 s so that much larger than that of the ordinary homogeneous materials (on the order of $10^{-14}-10^{-8}$ s)[24,25] TWMBT usually gives different predictions from Pennes model on the thermal behavior of living tissues when the time scale of interest is no more than the order of a few seconds.

While the CV relation only takes account of the fast-transient effects, the dual-phase-lagging (DPL) constitutive relation includes the micro-structural interactions as well as the fast-transient effects[26,27]

$$\mathbf{q}(\mathbf{r},t+\tau_q)=-k\nabla T(\mathbf{r},t+\tau_T).\tag{1.7}$$

According to this relation, the temperature gradient at a point $\mathbf{r}$ of the material at time $t+\tau_T$ corresponds to the heat flux density vector at time $t+\tau_q$. The delay time τ_T is interpreted as being caused by the micro-structural interactions, such as phonon-electron interaction or phonon scattering, and is called the phase-lag of the temperature gradient. In some literature, the first-order Taylor expansion of Eq. (1.7) is also called the DPL constitutive relation

$$\mathbf{q}(\mathbf{r},t)+\tau_q\frac{\partial\mathbf{q}(\mathbf{r},t)}{\partial t}=-k\left\{\nabla T(\mathbf{r},t)+\tau_T\frac{\partial}{\partial t}[\nabla T(\mathbf{r},t)]\right\}.\tag{1.8}$$

Combining Eq. (1.8) with the equation of energy conservation, Eq. (1.1), results in the DPL model of bioheat transfer, in which the source terms are also usually treated in the same way as Pennes model:[5,21,28]

$$\begin{aligned}&\left[1+\tau_q\frac{(\rho c\omega)^b_{mac}}{(\rho c)_{mac}}\right]\frac{\partial T_{mac}}{\partial t}+\tau_q\frac{\partial^2 T_{mac}}{\partial t^2}\\&=\alpha_{mac}\nabla^2T_{mac}+\alpha_{mac}\tau_T\frac{\partial}{\partial t}\left(\nabla^2T_{mac}\right)\\&+\frac{(\rho c\omega)^b_{mac}}{(\rho c)_{mac}}\left(T^a_{mac}-T_{mac}\right)+\frac{1}{(\rho c)_{mac}}\left(1+\tau_q\frac{\partial}{\partial t}\right)\\&\times\left(q_{m,mac}+q_{e,mac}\right).\end{aligned}\tag{1.9}$$

This equation is parabolic when $\tau_q < \tau_T$, and thus predicts a nonwavelike heat conduction that differs from the usual diffusion predicted by the classical parabolic heat conduction equation (1.3). When $\tau_q > \tau_T$, however, Eq. (1.9) predominantly predicts wavelike thermal signals. Also note that the DPL heat conduction equation (1.9) reduces to Pennes model when $\tau_q = \tau_T = 0$, and the hyperbolic bioheat model when $\tau_q > \tau_T = 0$. Moreover, the second-order DPL constitutive relations can be obtained by retaining up to the second order Taylor expansions for $\mathbf{q}$, T, or both, in Eq. (1.7).[26] The readers are referred to Refs. 21, 23, and 29 for the DPL models of bioheat transfer based on the higher order DPL constitutive relations. It has been shown that the DPL models can predict significantly different thermal behavior in magnetic hyperthermia, laser-irradiation and skin bioheat transfer processes from both the TWMBT and Fourier-type Pennes models.[5,21,23,28,30]

Basically, solving of the continuum models requires reliable data about the macroscale properties of the continuum, such as thermal conductivity, thermal diffusivity, perfusion rate, metabolic heat generation, and phase lags of the heat flux and the temperature gradient. Determination of these thermal properties is a technically challenging task, however, due to (i) the strong coupling among different heat transfer mechanisms in the tissues, (ii) the complex mechanical and thermal interactions between the instrument and tissue, (iii) the property nonuniformity due to the tissue heterogeneity, (iv) the significant sample-to-sample variability, and (v) the high sensitivity of tissue properties to outside influence.[2] For the systems containing large vessels which cannot be regarded as a part of the continuum, Chen and Holmes suggested that they should be modeled on an individual basis.[11]

Arteries and veins often appear as closely spaced pairs with different flow directions and temperatures.[31] Although there is no direct mass transfer between a countercurrent vessel pair, there can be an energy transfer through the heat conduction between arteries and tissue, and between veins and tissue.[31] Models were then developed by taking this energy transfer (and possibly pulsating countercurrent due to heart beating) into consideration.[31–41] The effect of vascular geometry (e.g., diameter, number density, and flow direction) has been involved in these more advanced models.

Although the early continuum-based bioheat models provide a usable approach for quantified prediction of the thermal behavior in biological tissues, they have not taken some important factors into account and involve many assumptions, therefore have moderate application range and accuracy. Note that the biological tissues can be regarded as blood-saturated porous media with extravascular tissues being the solid matrix. Several groups have applied porous media approach for developing bioheat models, because it has the potential to well reflect the influence of microscale physics on macroscale properties with minimized assumptions involved.[42–44] As the most advanced one, Nakayama and Kuwahara's[44] two-equation macroscopic bioheat model has a form of

In the blood phase

$$\begin{aligned}(\rho c\varepsilon)^b_{mac}&\left(\frac{\partial\langle T^b_{mic}\rangle^b}{\partial t}+\mathbf{v}^b_{mac}\cdot\nabla\langle T^b_{mic}\rangle^b\right)\\ &=\nabla\cdot\left[\left(\langle\varepsilon\mathbf{k}\rangle^b_{mac}+\varepsilon^b\mathbf{k}_{dis}\right)\cdot\nabla\langle T^b_{mic}\rangle^b\right]\\ &\quad-ha_v\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right)\\ &\quad-(\rho c\omega)^b_{mac}\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right)\end{aligned}\tag{1.10}$$

In the tissue phase

$$\begin{aligned}(\rho c\varepsilon)^t_{mac}\frac{\partial\langle T^t_{mic}\rangle^t}{\partial t}&=\nabla\cdot\left[(\varepsilon\mathbf{k})^t_{mac}\cdot\nabla\langle T^t_{mic}\rangle^t\right]\\ &\quad+ha_v\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right)\\ &\quad+(\rho c\omega)^b_{mac}\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right)\\ &\quad+(\varepsilon q_m)^t_{mac}\end{aligned}\tag{1.11}$$

where $\langle T^b_{mic}\rangle^b$ and $\langle T^t_{mic}\rangle^t$ are the intrinsic average temperature of blood and tissue phases, respectively [also see Eq. (2.14)]. ε is the volume fraction. $\mathbf{v}$ is the velocity. $\mathbf{k}$ and $\mathbf{k}_{dis}$ are the thermal conductivity tensor and dispersion thermal conductivity tensor, respectively. h and a_v are the interfacial heat transfer coefficient and specific surface area, respectively. This model considers local thermal nonequilibrium between the blood and the surrounding tissue, and includes the effects of blood-tissue conduction, convection, blood perfusion, and metabolic heat generation. The effect of microscale vascular structure is not fully considered in this model since a rigorous closure of the energy equations was not provided. Note that in the blood-phase equation (1.10), the macroscopic convection term (the second term on the left side), thermal dispersion term ($\mathbf{k}_{dis}$-involved term on the right side) and perfusion term (the last term on the right side) all come from the averaging of the convection term on microscopic energy equation for blood. As such, in the tissue-phase equation (1.11), it seems that the perfusion term (the third term on the right side) should not appear, just as the other two do not.

Arterial-venous anastomosis, a direct connection between an artery and a vein to bypass capillaries, occurs normally in fingers, nose and lip for the functions like blood flow rate adjustment and body temperature regulation.[45,46] To account for the countercurrent heat transfer between closely spaced arteries and veins induced by arterial-venous anastomoses, the two-equation model has also been extended to a three-equation version by Nakayama and Kuwahara.[44]

By decoupling Eqs. (1.10) and (1.11) to obtain the two energy equations with $\langle T^b_{mic}\rangle^b$ and $\langle T^t_{mic}\rangle^t$ as the sole unknown variable, respectively, we can find that both $\langle T^b_{mic}\rangle^b$ and $\langle T^t_{mic}\rangle^t$ are governed by DPL heat conduction equations, although the heat conduction in blood and tissue has been assumed to be Fourier-type at the microscale.[47] It thus confirms that the DPL constitutive relation is more proper for the heat flux density vector in developing macroscale bioheat models using the continuum approach. The formulas of τ_q and τ_T are also available for these new DPL heat conduction equations. Since the cross-coupling thermal dispersion is not considered in this model, the bioheat transfer is always diffusion dominant without thermal waves.

Compared with the continuum models, the model based on porous media theory offers the connection between microscale and macroscale properties and is capable to accurately describe the rich blood-tissue interaction. A rigorous closure theory is nevertheless needed for materializing its promising potential. In the following, we develop a closed macroscale bioheat model with blood or tissue temperature as the sole unknown variable, showing: (i) the DPL bioheat transport at macroscale for both blood and tissue phases, (ii) the sophisticated effects of the interfacial convective heat transfer, the blood velocity, the blood perfusion and the metabolic heat generation on macroscale temperature fields in both blood and tissue, and (iii) the possible thermal waves and resonance predicted by the bioheat transfer equations. Because of the significance of DPL heat conduction in bioheat problems, we subsequently present the solution structure theorems for mixed problems and Cauchy problems of DPL heat conduction equations. Finally, we apply the solution structure theorems to study the bioheat transport in skin tissue and during magnetic hyperthermia.

II. A GENERAL BIOHEAT MODEL AT MACROSCALE

In developing the macroscale bioheat model by the porous media approach, microscale field equations are first averaged over a representative elementary volume (REV) to obtain the macroscale field equations. Multiscale theorems are used in the averaging process to convert integrals of gradient, divergence, curl, and partial time derivatives of a function into some combination of gradient, divergence, curl, and partial time derivatives of integrals of the function and integrals over the boundary of the REV.[48] The closure models are then provided for the unclosed terms in macroscale field equations that represent the microscale effect to form a closed system.

A. Volume averaging

1. Microscale model

Neglect the gravitational effect and assume that blood is incompressible and Newtonian. By the conservation of mass, momentum and energy, and the Fourier's law of heat conduction, the microscale model for blood flow and heat conduction in the two phases read:

In the blood phase,

$$\nabla \cdot \mathbf{v}^b_{mic} = 0, \tag{2.1}$$

$$\rho^b_{mic}\frac{\partial \mathbf{v}^b_{mic}}{\partial t} + \rho^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\mathbf{v}^b_{mic} = -\nabla p^b_{mic} + \mu^b_{mic}\nabla^2\mathbf{v}^b_{mic}, \tag{2.2}$$

$$(\rho c)^b_{mic}\frac{\partial T^b_{mic}}{\partial t} + (\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla T^b_{mic} = \nabla\cdot\left(k^b_{mic}\nabla T^b_{mic}\right). \tag{2.3}$$

In the tissue phase,

$$(\rho c)^t_{mic}\frac{\partial T^t_{mic}}{\partial t} = \nabla\cdot\left(k^t_{mic}\nabla T^t_{mic}\right) + \Phi^t_{mic}. \tag{2.4}$$

Boundary conditions at the blood-tissue interface A_{bt}:

$$\text{B.C.1 } \mathbf{v}^b_{mic} = \mathbf{v}^b_{mic}\big|_{A_{bt}}, \tag{2.5}$$

$$\text{B.C.2 } T^b_{mic} = T^t_{mic}, \tag{2.6}$$

$$\text{B.C.3. } \mathbf{n}_{bt}\cdot k^b_{mic}\nabla T^b_{mic} = \mathbf{n}_{bt}\cdot k^t_{mic}\nabla T^t_{mic} + \Omega. \tag{2.7}$$

Here the subscript *mic* is used to indicate the microscale properties. p is the pressure, and μ the viscosity. Φ^t_{mic} is the homogeneous thermal source which may come from the metabolic heat generation or the external heat supply like the one used in hyperthermia. $\mathbf{n}_{bt}$ is the outward-directed surface normal vector from the b-phase toward the t-phase, and $\mathbf{n}_{bt} = -\mathbf{n}_{tb}$ (Fig. 1). Ω is the heterogeneous thermal source relating to the mass transfer at blood-tissue interfaces due to the blood perfusion. Assume that once the blood enters the capillaries from arterioles, it immediately equilibrates thermally with the tissue until it leaves the capillaries into venules. Ω thus satisfies

$$\Omega = \begin{cases} \mathbf{n}_{bt}\cdot(\rho c)^b_{mic}\mathbf{v}^b_{mic}\left(T^b_{mic} - T^t_{mic}\right), & \text{when } \mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic} > 0 \\ 0, \quad \text{when } \mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic} < 0 \end{cases} \tag{2.8}$$

2. Scaling-up by volume averaging

The system is assumed to be rigid so that V^b and V^t are time-independent. By integrating Eqs. (2.1)–(2.4) over the

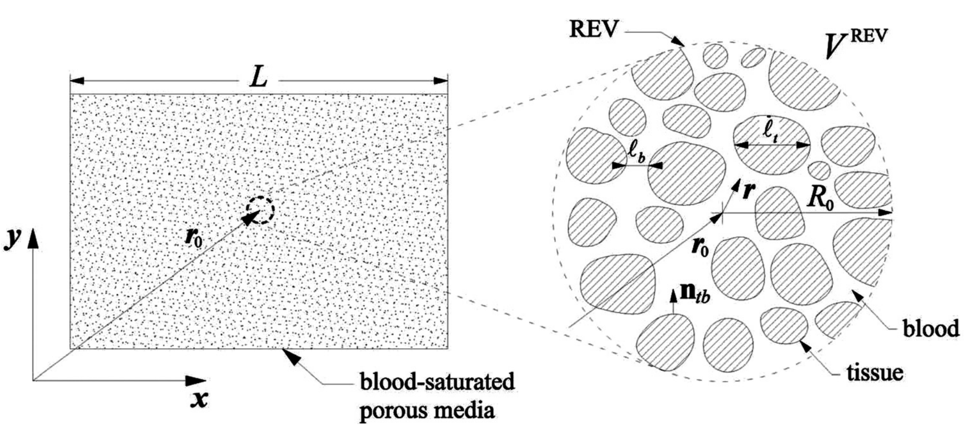

FIG. 1. Blood-saturated porous media and REV.

volumes of blood (V^b) or tissue (V^t) in the REV, we can have the superficial average form of the conservative equations of mass, momentum, and energy,

$$\left\langle \nabla \cdot \mathbf{v}^b_{mic} \right\rangle = 0, \tag{2.9}$$

$$\rho^b_{mac} \frac{\partial \left\langle \mathbf{v}^b_{mic} \right\rangle}{\partial t} + \rho^b_{mac} \left\langle \mathbf{v}^b_{mic} \cdot \nabla \mathbf{v}^b_{mic} \right\rangle = -\left\langle \nabla p^b_{mic} \right\rangle + \left\langle \mu^b_{mic} \nabla^2 \mathbf{v}^b_{mic} \right\rangle, \tag{2.10}$$

$$(\rho c)^b_{mac} \frac{\partial \left\langle T^b_{mic} \right\rangle}{\partial t} + (\rho c)^b_{mac} \left\langle \mathbf{v}^b_{mic} \cdot \nabla T^b_{mic} \right\rangle = \left\langle \nabla \cdot \left(k^b_{mic} \nabla T^b_{mic} \right) \right\rangle, \tag{2.11}$$

$$(\rho c)^t_{mac} \frac{\partial \left\langle T^t_{mic} \right\rangle}{\partial t} = \left\langle \nabla \cdot \left(k^t_{mic} \nabla T^t_{mic} \right) \right\rangle + \left\langle \Phi^t_{mic} \right\rangle, \tag{2.12}$$

where the notation $\langle\ \rangle$ indicates superficial average quantities as

$$\left\langle \psi^b \right\rangle = \frac{1}{V^{\rm REV}} \int_{V^b} \psi^b dV, \left\langle \psi^t \right\rangle = \frac{1}{V^{\rm REV}} \int_{V^t} \psi^t dV \tag{2.13}$$

The intrinsic averages relate to the superficial averages by

$$\left\langle \psi^b \right\rangle^b = \frac{1}{V^b} \int_{V^b} \psi^b dV = \frac{1}{\varepsilon^b_{mac}} \left\langle \psi^b \right\rangle, \quad \left\langle \psi^t \right\rangle^t = \frac{1}{V^t} \int_{V^t} \psi^t dV = \frac{1}{\varepsilon^t_{mac}} \left\langle \psi^t \right\rangle \tag{2.14}$$

Continuity equation:

Applying the spatial averaging theorem to Eq. (2.9) leads to

$$\left\langle \nabla \cdot \mathbf{v}^b_{mic} \right\rangle = \nabla \cdot \left\langle \mathbf{v}^b_{mic} \right\rangle + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot \mathbf{v}^b_{mic} dA = 0. \tag{2.15}$$

The surface integral represents the volumetric rate of blood bleeding off to the tissue phase through the vascular wall. Since the net filtration of blood from the intravascular to the extravascular regions is very small, the surface integral is negligible so that

$$\nabla \cdot \left\langle \mathbf{v}^b_{mic} \right\rangle = 0. \tag{2.16}$$

The superficial average velocity field is a preferred representation of the macroscale velocity field because it is solenoidal.[49]

Energy equation:

Applying Eq. (2.1) and the spatial average theorem, the second term in the left side of Eq. (2.11) can be written as

$$(\rho c)^b_{mac} \left\langle \mathbf{v}^b_{mic} \cdot \nabla T^b_{mic} \right\rangle = (\rho c)^b_{mac} \left(\nabla \cdot \left\langle \mathbf{v}^b_{mic} T^b_{mic} \right\rangle + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot \mathbf{v}^b_{mic} T^b_{mic} dA \right). \tag{2.17}$$

Decompose the microscale quantity into its intrinsic average quantity and a spatial deviation as

$$\mathbf{v}^b_{mic} = \left\langle \mathbf{v}^b_{mic} \right\rangle^b + \tilde{\mathbf{v}}^b_{mic}, T^b_{mic} = \left\langle T^b_{mic} \right\rangle^b + \tilde{T}^b_{mic}. \tag{2.18}$$

Therefore, the term $\left\langle \mathbf{v}^b_{mic} T^b_{mic} \right\rangle$ in Eq. (2.17) can be written as

$$\left\langle \mathbf{v}^b_{mic} T^b_{mic} \right\rangle = \left\langle \left\langle \mathbf{v}^b_{mic} \right\rangle^b \left\langle T^b_{mic} \right\rangle^b + \tilde{\mathbf{v}}^b_{mic} \left\langle T^b_{mic} \right\rangle^b + \left\langle \mathbf{v}^b_{mic} \right\rangle^b \tilde{T}^b_{mic} + \tilde{\mathbf{v}}^b_{mic} \tilde{T}^b_{mic} \right\rangle. \tag{2.19}$$

The second and third terms on the right side of Eq. (2.19) is negligible[50] so that it reduces to

$$\left\langle \mathbf{v}^b_{mic} T^b_{mic} \right\rangle = \varepsilon^b_{mac} \left\langle \mathbf{v}^b_{mic} \right\rangle^b \left\langle T^b_{mic} \right\rangle^b + \left\langle \tilde{\mathbf{v}}^b_{mic} \tilde{T}^b_{mic} \right\rangle. \tag{2.20}$$

The surface integral in Eq. (2.17) can be approximated by using Eq. (2.18)

$$\frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot \mathbf{v}^b_{mic} T^b_{mic} dA = \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot \mathbf{v}^b_{mic} \tilde{T}^b_{mic} dA. \tag{2.21}$$

For the right-side term in Eq. (2.11), it can be expanded by using Eq. (2.14), Eq. (2.18), and the spatial average theorem

$$\left\langle \nabla \cdot \left(k^b_{mic} \nabla T^b_{mic} \right) \right\rangle = \nabla \cdot \left[k^b_{mac} \left(\varepsilon^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \tilde{T}^b_{mic} dA \right) \right] + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot k^b_{mic} \nabla \left\langle T^b_{mic} \right\rangle^b dA + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot k^b_{mic} \nabla \tilde{T}^b_{mic} dA. \tag{2.22}$$

When the standard length scale constraints are valid (ℓ_b, $\ell_t \ll R_0$ and $R_0 \ll L$, where ℓ_b, ℓ_t, R_0, and L are the microscale length scales of the b- and t-phases, the radius of the REV and the system length scale, respectively; Fig. 1), we can neglect the term $\frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot k^b_{mic} \nabla \left\langle T^b_{mic} \right\rangle^b dA$ since[51]

$$\begin{aligned} \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot k^b_{mic} \nabla \left\langle T^b_{mic} \right\rangle^b dA &= \left\{ \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} dA \right\} \cdot k^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b \\ &= \frac{1}{V^{\rm REV}} \left(\int_{V^b} \nabla 1 dV - \nabla \int_{V^b} 1 dV \right) \cdot k^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b \\ &= -\nabla \left(\frac{1}{V^{\rm REV}} \int_{V^b} 1 dV \right) \cdot k^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b \\ &= -\nabla \varepsilon^b_{mac} \cdot k^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b \approx 0. \end{aligned} \tag{2.23}$$

Substituting Eqs. (2.20)–(2.23) into Eq. (2.11), we obtain the unclosed form of energy equation for b-phase

$$\begin{aligned} &\underbrace{(\rho c \varepsilon)^b_{mac} \frac{\partial \left\langle T^b_{mic} \right\rangle^b}{\partial t}}_{\text{accumulation}} + \underbrace{(\rho c \varepsilon)^b_{mac} \left\langle \mathbf{v}^b_{mic} \right\rangle^b \cdot \nabla \left\langle T^b_{mic} \right\rangle^b}_{\text{convection}} \\ &= \underbrace{\nabla \cdot \left[k^b_{mac} \left(\varepsilon^b_{mac} \nabla \left\langle T^b_{mic} \right\rangle^b + \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \tilde{T}^b_{mic} dA \right) \right]}_{\text{conduction}} \\ &\quad \underbrace{- (\rho c)^b_{mac} \nabla \cdot \left\langle \tilde{\mathbf{v}}^b_{mic} \tilde{T}^b_{mic} \right\rangle}_{\text{dispersion}} + \underbrace{\frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot k^b_{mic} \nabla \tilde{T}^b_{mic} dA}_{\text{interfacial flux}} \\ &\quad \underbrace{- (\rho c)^b_{mac} \frac{1}{V^{\rm REV}} \int_{A_{bt}} \mathbf{n}_{bt} \cdot \mathbf{v}^b_{mic} \tilde{T}^b_{mic} dA.}_{\text{blood perfusion}} \end{aligned} \tag{2.24}$$

The energy equation for t-phase can be analogously obtained

$$\underbrace{(\rho c\varepsilon)^t_{mac}\frac{\partial\langle T^t_{mic}\rangle^t}{\partial t}}_{\text{accumulation}} = \underbrace{\nabla\cdot\left[k^t_{mac}\left(\varepsilon^t_{mac}\nabla\langle T^b_{mic}\rangle^t + \frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}\tilde{T}^t_{mic}dA\right)\right]}_{\text{conduction}} + \underbrace{\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}\cdot k^t_{mic}\nabla\tilde{T}^t_{mic}dA}_{\text{interfacial flux}} + \underbrace{\varepsilon^t_{mac}\langle\Phi^t_{mic}\rangle^t}_{\substack{\text{metabolic}\\ \text{thermal source}}}, \tag{2.25}$$

where

$$T^t_{mic} = \langle T^t_{mic}\rangle^t + \tilde{T}^t_{mic}. \tag{2.26}$$

Momentum equation:

The momentum equation has a similar form with the b-phase energy equation[49]

$$\underbrace{(\rho\varepsilon)^b_{mac}\frac{\partial\langle \mathbf{v}^b_{mic}\rangle^b}{\partial t}}_{\text{accumulation}} + \underbrace{(\rho\varepsilon)^b_{mac}\langle \mathbf{v}^b_{mic}\rangle^b\cdot\nabla\langle \mathbf{v}^b_{mic}\rangle^b}_{\text{convection}} = \underbrace{-\varepsilon^b_{mac}\nabla\langle p^b_{mic}\rangle^b - \frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\tilde{p}^b_{mic}\text{dA}}_{\text{pressure force}} + \underbrace{\nabla\cdot\left[\mu^b_{mac}\left(\varepsilon^b_{mac}\nabla\langle \mathbf{v}^b_{mic}\rangle^b\right)\right]}_{\text{viscous force}} \underbrace{-\rho^b_{mac}\nabla\cdot\langle\tilde{\mathbf{v}}^b_{mic}\tilde{\mathbf{v}}^b_{mic}\rangle}_{\text{dispersion}} + \underbrace{\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mu^b_{mic}\nabla\tilde{\mathbf{v}}^b_{mic}\text{dA}}_{\text{interfacial viscous force}}, \tag{2.27}$$

where

$$p^b_{mic} = \langle p^b_{mic}\rangle^b + \tilde{p}^b_{mic}. \tag{2.28}$$

The closed macroscale transport model requires the closure models for spatial deviation variables ($\tilde{\mathbf{v}}^b_{mic}$, $\tilde{p}^b_{mic}$, $\tilde{T}^b_{mic}$, and $\tilde{T}^t_{mic}$). Readers are referred to Ref. 52 for the closure details regarding $\tilde{\mathbf{v}}^b_{mic}$ and $\tilde{p}^b_{mic}$ for the case of homogeneous porous media, which lead to the Forchheimer equation. When Reynolds number of the blood flow is smaller than one (such as the blood flow in small vessels), Darcy's law can be used as the macroscale momentum equation.[44,49]

3. Energy equation closure

In order to obtain a closed system, closure models need to be provided for $\tilde{T}^b_{mic}$ and $\tilde{T}^t_{mic}$ in Eqs. (2.24) and (2.25), respectively. Apply the decompositions given by Eqs. (2.18) and (2.26) into Eqs. (2.3) and (2.4), and then subtract the results from Eqs. (2.24) and (2.25) with both sides divided by ε^b_{mac} and ε^t_{mac}, respectively. We obtain

$$\underbrace{(\rho c)^b_{mic}\frac{\partial\tilde{T}^b_{mic}}{\partial t}}_{\text{accumulation}} + (\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\tilde{T}^b_{mic} + (\rho c)^b_{mic}\tilde{\mathbf{v}}^b_{mic}\cdot\nabla\langle T^b_{mic}\rangle^b = \nabla\cdot\left(k^b_{mic}\nabla\tilde{T}^b_{mic}\right) \underbrace{-\frac{1}{\varepsilon^b_{mac}}\nabla\cdot\left(\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}k^b_{mic}\tilde{T}^b_{mic}dA\right)}_{\text{non-local conduction}} + \underbrace{(\rho c)^b_{mac}\nabla\cdot\langle\tilde{\mathbf{v}}^b_{mic}\tilde{T}^b_{mic}\rangle^b}_{\text{non-local convection}} - \frac{1}{\varepsilon^b_{mac}}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic} \times\nabla\tilde{T}^b_{mic}dA + \left(\frac{\rho c}{\varepsilon}\right)^b_{mac}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\tilde{T}^b_{mic}dA, \tag{2.29}$$

$$\underbrace{(\rho c)^t_{mic}\frac{\partial\tilde{T}^t_{mic}}{\partial t}}_{\text{accumulation}} = \nabla\cdot\left(k^t_{mic}\nabla\tilde{T}^t_{mic}\right) \underbrace{-\frac{1}{\varepsilon^t_{mac}}\nabla\cdot\left(\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}k^t_{mic}\tilde{T}^t_{mic}dA\right)}_{\text{non-local conduction}} - \frac{1}{\varepsilon^t_{mac}}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}\cdot k^t_{mic}\nabla\tilde{T}^t_{mic}dA. \tag{2.30}$$

When the length scale restriction $\ell_b,\ \ell_t \ll L$ is satisfied, the non-local conduction and non-local convection terms can be neglected because[53]

$$\frac{1}{\varepsilon^b_{mac}}\nabla\cdot\left(\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}k^b_{mic}\tilde{T}^b_{mic}dA\right) \ll \nabla\cdot\left(k^b_{mic}\nabla\tilde{T}^b_{mic}\right), \tag{2.31}$$

$$\frac{1}{\varepsilon^t_{mac}}\nabla\cdot\left(\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}k^t_{mic}\tilde{T}^t_{mic}dA\right) \ll \nabla\cdot\left(k^t_{mic}\nabla\tilde{T}^t_{mic}\right), \tag{2.32}$$

$$(\rho c)^b_{mac}\nabla\cdot\langle\tilde{\mathbf{v}}^b_{mic}\tilde{T}^b_{mic}\rangle^b \ll (\rho c)^b_{mac}\mathbf{v}^b_{mic}\cdot\nabla\tilde{T}^b_{mic}. \tag{2.33}$$

We can further neglect the accumulation terms in Eqs. (2.29) and (2.30) for a quasisteady closure. Therefore, the governing equations for $\tilde{T}^b_{mic}$ and $\tilde{T}^t_{mic}$ reduce to

$$(\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\tilde{T}^b_{mic} + (\rho c)^b_{mic}\tilde{\mathbf{v}}^b_{mic}\cdot\nabla\langle T^b_{mic}\rangle^b = \nabla\cdot\left(k^b_{mic}\nabla\tilde{T}^b_{mic}\right) - \frac{1}{\varepsilon^b_{mac}}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\tilde{T}^b_{mic}dA + \left(\frac{\rho c}{\varepsilon}\right)^b_{mac}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\tilde{T}^b_{mic}dA, \tag{2.34}$$

$$0 = \nabla\cdot\left(k^t_{mic}\nabla\tilde{T}^t_{mic}\right) - \frac{1}{\varepsilon^t_{mac}}\frac{1}{V\text{REV}}\int_{A_{bt}}\mathbf{n}_{tb}\cdot k^t_{mic}\nabla\tilde{T}^t_{mic}dA. \tag{2.35}$$

Boundary conditions:

$$\text{B.C.1}\ \tilde{\mathrm{T}}^b_{mic} = \tilde{\mathrm{T}}^t_{mic} - \left(\langle \mathrm{T}^b_{mic}\rangle^b - \langle \mathrm{T}^t_{mic}\rangle^t\right),\ \text{at}\ A_{bt}, \tag{2.36}$$

$$\text{B.C.2}\ \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\tilde{\mathrm{T}}^b_{mic} = \mathbf{n}_{bt}\cdot k^t_{mic}\nabla\tilde{\mathrm{T}}^t_{mic} - \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\langle \mathrm{T}^b_{mic}\rangle^b + \mathbf{n}_{bt}\cdot k^t_{mic}\nabla\langle \mathrm{T}^t_{mic}\rangle^t + \Omega,\ \text{at}\ A_{bt}. \tag{2.37}$$

Average condition:

$$\left\langle \tilde{T}^b_{mic} \right\rangle^b = 0, \left\langle \tilde{T}^t_{mic} \right\rangle^t = 0. \quad (2.38)$$

Periodicity condition:

$$\tilde{T}^b_{mic}(\mathbf{r}+\ell_i) = \tilde{T}^b_{mic}(\mathbf{r}), \tilde{T}^t_{mic}(\mathbf{r}+\ell_i) = \tilde{T}^t_{mic}(\mathbf{r}), i = 1,2,3. \quad (2.39)$$

Here, $\mathbf{r}$ is a position vector and $\ell_i (i = 1,2,3)$ represents the lattice vectors. The periodicity condition is imposed because the closure problem is normally solved only in some representative region that can be treated as a *unit cell* in a spatially periodic model.[50] The unit cell should contain enough information regarding the microstructure of the biological tissues in order to be representative of the real system. Specific anatomy like vascular countercurrent structure can also be reflected by properly constructing the unit cell. The average condition in Eq. (2.38) is required to determine the surface integrals in Eqs. (2.34) and (2.35). These two-surface integrals are related by, based on the flux boundary condition (2.37),

$$\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}} \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\tilde{T}^b_{mic}dA = \frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}} \mathbf{n}_{bt}\cdot k^t_{mic}\nabla\tilde{T}^t_{mic}dA + a_v\langle\Omega\rangle^{bt}_{mac}, \quad (2.40)$$

where

$$\langle\Omega\rangle^{bt}_{mac} = \frac{1}{A_{bt}}\int_{A_{at}}\Omega dA. \quad (2.41)$$

The A_{bt} in Eq. (2.41) denotes the blood-tissue interfacial area in V^{REV}, and A_{at} indicates that Ω is integrated over the arteriole-tissue interfaces since $\Omega = 0$ at venule-tissue interfaces [Eq. (2.8)]. $\langle\Omega\rangle^{bt}_{mac}$ gives rise to coupling at the macroscopic level and can be modeled by

$$\langle\Omega\rangle^{bt}_{mac} = \frac{1}{a_v}(\rho c\omega)^b_{mac}\left(\left\langle T^b_{mic}\right\rangle^b - \left\langle T^t_{mic}\right\rangle^t\right). \quad (2.42)$$

We view $\nabla\left\langle \mathrm{T}^b_{mic}\right\rangle^b$, $\nabla\left\langle \mathrm{T}^t_{mic}\right\rangle^t$, $\left\langle \mathrm{T}^b_{mic}\right\rangle^b - \left\langle \mathrm{T}^t_{mic}\right\rangle^t$, and $\langle\Omega\rangle^{bt}_{mac}$ as the sources for spatial deviation temperatures so that[47,49]

$$\tilde{\mathrm{T}}^b_{mic} = \mathbf{b}^{bb}_{mic}\cdot\nabla\left\langle \mathrm{T}^b_{mic}\right\rangle^b + \mathbf{b}^{bt}_{mic}\cdot\nabla\left\langle \mathrm{T}^t_{mic}\right\rangle^t - s^b_{mic}\left(\left\langle \mathrm{T}^b_{mic}\right\rangle^b - \left\langle \mathrm{T}^t_{mic}\right\rangle^t\right) + r^b_{mic}\langle\Omega\rangle^{bt}_{mac}, \quad (2.43)$$

$$\tilde{\mathrm{T}}^t_{mic} = \mathbf{b}^{tb}_{mic}\cdot\nabla\left\langle \mathrm{T}^b_{mic}\right\rangle^b + \mathbf{b}^{tt}_{mic}\cdot\nabla\left\langle \mathrm{T}^t_{mic}\right\rangle^t - s^t_{mic}\left(\left\langle \mathrm{T}^b_{mic}\right\rangle^b - \left\langle \mathrm{T}^t_{mic}\right\rangle^t\right) + r^t_{mic}\langle\Omega\rangle^{bt}_{mac}. \quad (2.44)$$

Here $\mathbf{b}^{bb}_{mic}$, $\mathbf{b}^{bt}_{mic}$, $\mathbf{b}^{tb}_{mic}$, $\mathbf{b}^{tt}_{mic}$, s^b_{mic}, s^t_{mic}, r^b_{mic}, and r^t_{mic} are the closure variables or the mapping variables that link the microscale and macroscale. Substituting the expression of $\langle\Omega\rangle^{bt}_{mac}$ given by Eq. (2.42) into Eqs. (2.43) and (2.44), we obtain

$$\tilde{\mathrm{T}}^b_{mic} = \mathbf{b}^{bb}_{mic}\cdot\nabla\left\langle \mathrm{T}^b_{mic}\right\rangle^b + \mathbf{b}^{bt}_{mic}\cdot\nabla\left\langle \mathrm{T}^t_{mic}\right\rangle^t - \sigma^b_{mic}\left(\left\langle \mathrm{T}^b_{mic}\right\rangle^b - \left\langle \mathrm{T}^t_{mic}\right\rangle^t\right), \quad (2.45)$$

$$\tilde{\mathrm{T}}^t_{mic} = \mathbf{b}^{tb}_{mic}\cdot\nabla\left\langle \mathrm{T}^b_{mic}\right\rangle^b + \mathbf{b}^{tt}_{mic}\cdot\nabla\left\langle \mathrm{T}^t_{mic}\right\rangle^t - \sigma^t_{mic}\left(\left\langle \mathrm{T}^b_{mic}\right\rangle^b - \left\langle \mathrm{T}^t_{mic}\right\rangle^t\right), \quad (2.46)$$

where

$$\sigma^b_{mic} = s^b_{mic} - \frac{r^b_{mic}(\rho c\omega)^b_{mac}}{a_v}, \sigma^t_{mic} = s^t_{mic} - \frac{r^t_{mic}(\rho c\omega)^b_{mac}}{a_v}. \quad (2.47)$$

By substituting Eqs. (2.45) and (2.46) into the governing equations and boundary conditions for $\tilde{T}^b_{mic}$ and $\tilde{T}^t_{mic}$ [Eqs. (2.34)–(2.39)], we obtain the following three closure problems:

Problem I

$$(\rho c)^b_{mic}\tilde{\mathbf{v}}^b_{mic} + (\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\mathbf{b}^{bb}_{mic} = k^b_{mic}\nabla^2\mathbf{b}^{bb}_{mic} - \frac{1}{\varepsilon^b_{mac}}\mathbf{c}^{bb}_{mic}, \text{ in the } b\text{-phase}, \quad (2.48)$$

$$0 = k^t_{mic}\nabla^2\mathbf{b}^{tb}_{mic} + \frac{1}{\varepsilon^t_{mac}}\mathbf{c}^{tb}_{mic}, \text{ in the } t\text{-phase}, \quad (2.49)$$

$$\text{B.C.1 } \mathbf{b}^{bb}_{mic} = \mathbf{b}^{tb}_{mic}, \text{at } A_{bt}, \quad (2.50)$$

$$\text{B.C.2 } \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bb}_{mic} = \mathbf{n}_{bt}\cdot k^t_{mic}\nabla\mathbf{b}^{tb}_{mic} - \mathbf{n}_{bt}k^b_{mic}, \text{ at } A_{bt}. \quad (2.51)$$

Average:

$$\left\langle \mathbf{b}^{bb}_{mic}\right\rangle^b = 0, \left\langle \mathbf{b}^{tb}_{mic}\right\rangle^t = 0. \quad (2.52)$$

Periodicity:

$$\mathbf{b}^{bb}_{mic}(\mathbf{r}+\ell_i) = \mathbf{b}^{bb}_{mic}(\mathbf{r}), \mathbf{b}^{tb}_{mic}(\mathbf{r}+\ell_i) = \mathbf{b}^{tb}_{mic}(\mathbf{r}), \quad i = 1,2,3, \quad (2.53)$$

where

$$\mathbf{c}^{bb}_{mic} = \frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bb}_{mic}dA - (\rho c)^b_{mac}\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\mathbf{b}^{bb}_{mic}dA, \quad (2.54)$$

$$\mathbf{c}^{tb}_{mic} = \frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^t_{mic}\nabla\mathbf{b}^{tb}_{mic}dA. \quad (2.55)$$

Problem II

$$(\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\mathbf{b}^{bt}_{mic} = k^b_{mic}\nabla^2\mathbf{b}^{bt}_{mic} - \frac{1}{\varepsilon^b_{mac}}\mathbf{c}^{bt}_{mic}, \text{ in the } b\text{-phase}, \quad (2.56)$$

$$0 = k^t_{mic}\nabla^2\mathbf{b}^{bt}_{mic} + \frac{1}{\varepsilon^t_{mac}}\mathbf{c}^{tt}_{mic}, \text{ in the } t\text{-phase}. \quad (2.57)$$

$$\text{B.C.1 } \mathbf{b}^{bt}_{mic} = \mathbf{b}^{tt}_{mic}, \text{ at } A_{bt}, \quad (2.58)$$

$$\text{B.C.2 } \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bt}_{mic} = \mathbf{n}_{bt}\cdot k^t_{mic}\nabla\mathbf{b}^{tt}_{mic} + \mathbf{n}_{bt}k^t_{mac}, \text{ at } A_{bt}. \quad (2.59)$$

Average:

$$\left\langle \mathbf{b}^{bt}_{mic}\right\rangle^b = 0, \left\langle \mathbf{b}^{tt}_{mic}\right\rangle^t = 0. \quad (2.60)$$

Periodicity:

$$\mathbf{b}^{bt}_{mic}(\mathbf{r}+\ell_i)=\mathbf{b}^{bt}_{mic}(\mathbf{r}),\mathbf{b}^{tt}_{mic}(\mathbf{r}+\ell_i)=\mathbf{b}^{tt}_{mic}(\mathbf{r}),\quad i=1,2,3, \tag{2.61}$$

where

$$\mathbf{c}^{bt}_{mic}=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bt}_{mic}dA - (\rho c)^b_{mac}\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\mathbf{b}^{bt}_{mic}dA, \tag{2.62}$$

$$\mathbf{c}^{tt}_{mic}=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^t_{mic}\nabla\mathbf{b}^{tt}_{mic}dA. \tag{2.63}$$

Problem III

$$(\rho c)^b_{mic}\mathbf{v}^b_{mic}\cdot\nabla\sigma^b_{mic}=k^b_{mic}\nabla^2\sigma^b_{mic}-\frac{1}{\varepsilon^b_{mac}}G^b,\ \text{in the } b\text{-phase}, \tag{2.64}$$

$$0=k^t_{mic}\nabla^2\sigma^t_{mic}+\frac{1}{\varepsilon^t_{mac}}G^t,\ \text{in the } t\text{-phase}. \tag{2.65}$$

$$\text{B.C.1}\ \sigma^b_{mic}=\sigma^t_{mic}+1,\ \text{at } A_{bt}, \tag{2.66}$$

$$\text{B.C.2}\ \mathbf{n}_{bt}\cdot k^b_{mic}\nabla\sigma^b_{mic}=\mathbf{n}_{bt}\cdot k^t_{mic}\nabla\sigma^t_{mic}-\frac{(\rho c\omega)^b_{mac}}{a_v},\ \text{at } A_{bt}. \tag{2.67}$$

Average:

$$\langle\sigma^b_{mic}\rangle^b=0,\langle\sigma^t_{mic}\rangle^t=0. \tag{2.68}$$

Periodicity:

$$\sigma^b_{mic}(\mathbf{r}+\ell_i)=\sigma^b_{mic}(\mathbf{r}),\sigma^t_{mic}(\mathbf{r}+\ell_i)=\sigma^t_{mic}(\mathbf{r}),\quad i=1,2,3, \tag{2.69}$$

where

$$G^b=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\sigma^b_{mic}dA - (\rho c)^b_{mac}\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\sigma^b_{mic}dA, \tag{2.70}$$

$$G^t=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^t_{mic}\nabla\sigma^t_{mic}dA. \tag{2.71}$$

The three closure problems can be effectively resolved by standard numerical schemes. Readers are referred to, for example, Refs. 51 and 54 that solve some similar closure problems for heat transfer in porous media, and Refs. 55 and 56 for heat conduction in nanofluids.

The closed energy equations for b- and t-phases can be obtained by substituting Eqs. (2.45) and (2.46) into Eqs. (2.24) and (2.25).

For the b-phase:

$$\begin{aligned}&(\rho c\varepsilon)^b_{mac}\frac{\partial\langle T^b_{mic}\rangle^b}{\partial t}+(\rho c\varepsilon)^b_{mac}\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\langle T^b_{mic}\rangle^b\\&-\mathbf{u}^{bb}_{mac}\cdot\nabla\langle T^b_{mic}\rangle^b-\mathbf{u}^{bt}_{mac}\cdot\nabla\langle T^t_{mic}\rangle^t\\&=\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\langle T^b_{mic}\rangle^b+\mathbf{K}^{bt}_{mac}\cdot\nabla\langle T^t_{mic}\rangle^t\right)\\&-G^b\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right),\end{aligned} \tag{2.72}$$

in which the dominant thermal dispersion tensor $\mathbf{K}^{bb}_{mac}$ and the coupling thermal dispersion tensor $\mathbf{K}^{bt}_{mac}$ are given by

$$\mathbf{K}^{bb}_{mac}=(\varepsilon k)^b_{mac}\mathbf{I}+\frac{k^b_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\mathbf{b}^{bb}_{mic}dA-(\rho c)^b_{mac}\langle\tilde{\mathbf{v}}^b_{mic}\mathbf{b}^{bb}_{mic}\rangle, \tag{2.73}$$

$$\mathbf{K}^{bt}_{mac}=\frac{k^b_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\mathbf{b}^{bt}_{mic}dA-(\rho c)^b_{mac}\langle\tilde{\mathbf{v}}^b_{mic}\mathbf{b}^{bt}_{mic}\rangle. \tag{2.74}$$

The two nontraditional convective transport terms in Eq. (2.72) depend on the coefficients $\mathbf{u}^{bb}_{mac}$ and $\mathbf{u}^{bt}_{mac}$

$$\begin{aligned}\mathbf{u}^{bb}_{mac}=&\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bb}_{mic}dA\\&-\frac{k^b_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\sigma^b_{mic}dA+(\rho c)^b_{mac}\langle\tilde{\mathbf{v}}^b_{mic}\sigma^b_{mic}\rangle\\&-(\rho c)^b_{mac}\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\mathbf{b}^{bb}_{mic}dA,\end{aligned} \tag{2.75}$$

$$\begin{aligned}\mathbf{u}^{bt}_{mac}=&\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot k^b_{mic}\nabla\mathbf{b}^{bt}_{mic}dA\\&+\frac{k^b_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\sigma^b_{mic}dA-(\rho c)^b_{mac}\langle\tilde{\mathbf{v}}^b_{mic}\sigma^b_{mic}\rangle\\&-(\rho c)^b_{mac}\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{bt}\cdot\mathbf{v}^b_{mic}\mathbf{b}^{bt}_{mic}dA.\end{aligned} \tag{2.76}$$

For the t-phase:

$$\begin{aligned}&(\rho c\varepsilon)^t_{mac}\frac{\partial\langle T^t_{mic}\rangle^t}{\partial t}-\mathbf{u}^{tb}_{mac}\cdot\nabla\langle T^b_{mic}\rangle^b-\mathbf{u}^{tt}_{mac}\cdot\nabla\langle T^t_{mic}\rangle^t\\&=\nabla\cdot\left(\mathbf{K}^{tb}_{mac}\cdot\nabla\langle T^b_{mic}\rangle^b+\mathbf{K}^{tt}_{mac}\cdot\nabla\langle T^t_{mic}\rangle^t\right)\\&+G^t\left(\langle T^b_{mic}\rangle^b-\langle T^t_{mic}\rangle^t\right)+\varepsilon^t_{mac}\langle\Phi^t_{mic}\rangle^t,\end{aligned} \tag{2.77}$$

in which the coupling thermal conductivity tensor $\mathbf{K}^{tb}_{mac}$ and the effective thermal conductivity tensor $\mathbf{K}^{tt}_{mac}$ are given by

$$\mathbf{K}^{tb}_{mac}=\frac{k^t_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\mathbf{b}^{tb}_{mic}dA, \tag{2.78}$$

$$\mathbf{K}^{tt}_{mac}=(\varepsilon k)^t_{mac}\mathbf{I}+\frac{k^t_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\mathbf{b}^{tt}_{mic}dA. \tag{2.79}$$

The two velocitylike coefficients are given by

$$\mathbf{u}^{tb}_{mac}=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\cdot k^t_{mic}\nabla\mathbf{b}^{tb}_{mic}dA-\frac{k^t_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\sigma^t_{mic}dA, \tag{2.80}$$

$$\mathbf{u}^{tt}_{mac}=\frac{1}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\cdot k^t_{mic}\nabla\mathbf{b}^{tt}_{mic}dA+\frac{k^t_{mac}}{V^{\mathrm{REV}}}\int_{A_{bt}}\mathbf{n}_{tb}\sigma^t_{mic}dA. \tag{2.81}$$

B. Macroscale model for blood and tissue temperatures

Rewrite Eqs. (2.72) and (2.77) in their operator form

$$\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{C}&\mathbf{D}\end{bmatrix}\begin{bmatrix}\langle T^b_{mic}\rangle^b\\\langle T^t_{mic}\rangle^t\end{bmatrix}=\begin{bmatrix}0\\\varepsilon^t_{mac}\langle\Phi^t_{mic}\rangle^t\end{bmatrix}, \tag{2.82}$$

where

$$\mathbf{A}=\gamma^b_{mac}\left(\frac{\partial}{\partial t}+\left\langle \mathbf{v}^b_{mic}\right\rangle^b\cdot\nabla\right)-\mathbf{u}^{bb}_{mac}\cdot\nabla-\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\right)+G^b, \tag{2.83}$$

$$\mathbf{B}=-\mathbf{u}^{bt}_{mac}\cdot\nabla-\nabla\cdot\left(\mathbf{K}^{bt}_{mac}\cdot\nabla\right)-G^b, \tag{2.84}$$

$$\mathbf{C}=-\mathbf{u}^{tb}_{mac}\cdot\nabla-\nabla\cdot\left(\mathbf{K}^{tb}_{mac}\cdot\nabla\right)-G^t, \tag{2.85}$$

$$\mathbf{D}=\gamma^t_{mac}\frac{\partial}{\partial t}-\mathbf{u}^{tt}_{mac}\cdot\nabla-\nabla\cdot\left(\mathbf{K}^{tt}_{mac}\cdot\nabla\right)+G^t, \tag{2.86}$$

$$\gamma^b_{mac}=(\rho c\varepsilon)^b_{mac},\gamma^t_{mac}=(\rho c\varepsilon)^t_{mac}. \tag{2.87}$$

We then obtain the uncoupled form for $\langle T^b_{mic}\rangle^b$ and $\langle T^t_{mic}\rangle^t$ by multiplying the inverse matrix of $\left[\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right]$ with both sides of Eq. (2.82):

$$\begin{aligned}
&\frac{\partial\langle T^i_{mic}\rangle^i}{\partial t}+\frac{(\gamma^b\gamma^t)_{mac}}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\frac{\partial^2\langle T^i_{mic}\rangle^i}{\partial^2 t}+\frac{\gamma^b_{mac}G^t}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\langle T^i_{mic}\rangle^i\\
&\quad-\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\left\{G^b\left[\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)+\left(\mathbf{u}^{tb}_{mac}\cdot\nabla\right)\right]+G^t\left[\left(\mathbf{u}^{bb}_{mac}\cdot\nabla\right)+\left(\mathbf{u}^{bt}_{mac}\cdot\nabla\right)\right]\right\}\langle T^i_{mic}\rangle^i\\
&\quad=\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\left\{G^t\nabla\cdot\left[\left(\mathbf{K}^{bb}_{mac}+\mathbf{K}^{bt}_{mac}\right)\cdot\nabla\right]+G^b\nabla\cdot\left[\left(\mathbf{K}^{tb}_{mac}+\mathbf{K}^{tt}_{mac}\right)\cdot\nabla\right]\right\}\langle T^i_{mic}\rangle^i\\
&\qquad+\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\left\{\gamma^b_{mac}\frac{\partial}{\partial t}\left[\nabla\cdot\left(\mathbf{K}^{tt}_{mac}\cdot\nabla\right)\right]+\gamma^t_{mac}\frac{\partial}{\partial t}\left[\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\right)\right]\right\}\langle T^i_{mic}\rangle^i\\
&\qquad+\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\left\{\left[\nabla\cdot\left(\mathbf{K}^{bt}_{mac}\cdot\nabla\right)\right]\left[\nabla\cdot\left(\mathbf{K}^{tb}_{mac}\cdot\nabla\right)\right]-\left[\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\right)\right]\left[\nabla\cdot\left(\mathbf{K}^{tt}_{mac}\cdot\nabla\right)\right]\right.\\
&\qquad+\left[\gamma^b_{mac}\frac{\partial}{\partial t}\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)+\gamma^t_{mac}\frac{\partial}{\partial t}\left(\mathbf{u}^{bb}_{mac}\cdot\nabla\right)\right]-(\gamma^b\gamma^t)_{mac}\frac{\partial}{\partial t}\left(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\right)\\
&\qquad+\gamma^b_{mac}\left(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\right)\left[\nabla\cdot\left(\mathbf{K}^{tt}_{mac}\cdot\nabla\right)\right]+\gamma^b_{mac}\left(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\right)\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)-\left[\left(\mathbf{u}^{bb}_{mac}\cdot\nabla\right)\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)\right.\\
&\qquad\left.-\left(\mathbf{u}^{bt}_{mac}\cdot\nabla\right)\left(\mathbf{u}^{tb}_{mac}\cdot\nabla\right)\right]-\left\{\left(\mathbf{u}^{bb}_{mac}\cdot\nabla\right)\left[\nabla\cdot\left(\mathbf{K}^{tt}_{mac}\cdot\nabla\right)\right]+\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)\left[\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\right)\right]\right.\\
&\qquad\left.\left.-\left(\mathbf{u}^{bt}_{mac}\cdot\nabla\right)\left[\nabla\cdot\left(\mathbf{K}^{tb}_{mac}\cdot\nabla\right)\right]-\left(\mathbf{u}^{tb}_{mac}\cdot\nabla\right)\left[\nabla\cdot\left(\mathbf{K}^{bt}_{mac}\cdot\nabla\right)\right]\right\}\right\}\langle T^i_{mic}\rangle^i\\
&\qquad+\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\mathbf{H}^i\varepsilon^t_{mac}\langle\Phi^t_{mic}\rangle^t,
\end{aligned} \tag{2.88}$$

where the index i can take b or t. $\mathbf{H}^i$ takes the form

$$\mathbf{H}^b=\mathbf{u}^{bt}_{mac}\cdot\nabla+\nabla\cdot\left(\mathbf{K}^{bt}_{mac}\cdot\nabla\right)+G^b, \tag{2.89}$$

$$\begin{aligned}\mathbf{H}^t=\gamma^b_{mac}\left[\frac{\partial}{\partial t}+\left(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla\right)\right]-\mathbf{u}^{bb}_{mac}\cdot\nabla\\
-\nabla\cdot\left(\mathbf{K}^{bb}_{mac}\cdot\nabla\right)+G^b.\end{aligned} \tag{2.90}$$

When the system is isotropic and the physical properties of the two phases are constant, it reduces to

$$\begin{aligned}
&\frac{\partial\langle T^i_{mic}\rangle^i}{\partial t}+\tau_q\frac{\partial^2\langle T^i_{mic}\rangle^i}{\partial^2 t}+\frac{1}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}\\
&\times\left\{G^t\gamma^b_{mac}\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla-G^b\left[\left(\mathbf{u}^{tt}_{mac}\cdot\nabla\right)+\left(\mathbf{u}^{tb}_{mac}\cdot\nabla\right)\right]\right.\\
&\left.-G^t\left[\left(\mathbf{u}^{bb}_{mac}\cdot\nabla\right)+\left(\mathbf{u}^{bt}_{mac}\cdot\nabla\right)\right]\right\}\langle T^i_{mic}\rangle^i\\
&=\alpha\Delta\langle T^i_{mic}\rangle^i\\
&\quad+\alpha\tau_T\frac{\partial}{\partial t}\left(\Delta\langle T^i_{mic}\rangle^i\right)+\frac{\alpha}{k_e}\left[F(\mathbf{r},t)+\tau_q\frac{\partial F(\mathbf{r},t)}{\partial t}\right]^i_{mac},
\end{aligned} \tag{2.91}$$

where

$$\tau_q=\frac{(\gamma^b\gamma^t)_{mac}}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}, \tag{2.92}$$

$$\tau_T=\frac{(\gamma^b k^{tt}+\gamma^t k^{bb})_{mac}}{k_e}, \tag{2.93}$$

$$\alpha=\frac{k_e}{\rho c}=\frac{k_e}{G^b\gamma^t_{mac}+G^t\gamma^b_{mac}}, \tag{2.94}$$

$$k_e=G^t\left(k^{bb}_{mac}+k^{bt}_{mac}\right)+G^b\left(k^{tb}_{mac}+k^{tt}_{mac}\right), \tag{2.95}$$

$$\begin{aligned}
&\left[F(\mathbf{r},t)+\tau_q\frac{\partial F(\mathbf{r},t)}{\partial t}\right]^i_{mac}\\
&=\left\{(k^{bt}_{mac}k^{tb}_{mac}-k^{bb}_{mac}k^{tt}_{mac})\Delta^2+\left[\gamma^b_{mac}\frac{\partial}{\partial t}(\mathbf{u}^{tt}_{mac}\cdot\nabla)\right.\right.\\
&\quad\left.+\gamma^t_{mac}\frac{\partial}{\partial t}(\mathbf{u}^{bb}_{mac}\cdot\nabla)\right]-(\gamma^b\gamma^t)_{mac}\frac{\partial}{\partial t}(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla)\\
&\quad+\gamma^b_{mac}k^{tt}_{mac}\Delta(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla)+\gamma^b_{mac}(\langle\mathbf{v}^b_{mic}\rangle^b\cdot\nabla)(\mathbf{u}^{tt}_{mac}\cdot\nabla)\\
&\quad-[(\mathbf{u}^{bb}_{mac}\cdot\nabla)(\mathbf{u}^{tt}_{mac}\cdot\nabla)-(\mathbf{u}^{bt}_{mac}\cdot\nabla)(\mathbf{u}^{tb}_{mac}\cdot\nabla)]\\
&\quad-[k^{tt}_{mac}\Delta(\mathbf{u}^{bb}_{mac}\cdot\nabla)+k^{bb}_{mac}\Delta(\mathbf{u}^{tt}_{mac}\cdot\nabla)-k^{tb}_{mac}\Delta(\mathbf{u}^{bt}_{mac}\cdot\nabla)\\
&\quad\left.-k^{bt}_{mac}\Delta(\mathbf{u}^{tb}_{mac}\cdot\nabla)]\right\}\langle T^i_{mic}\rangle^i+h^i\varepsilon^t_{mac}\langle\Phi^t_{mic}\rangle^t,
\end{aligned} \tag{2.96}$$

$$h^b=\mathbf{u}^{bt}_{mac}\cdot\nabla+k^{bt}_{mac}\Delta+G^b, \tag{2.97}$$

$$h^t = \gamma^b_{mac}\left[\frac{\partial}{\partial t} + \left(\left\langle \mathbf{v}^b_{mic}\right\rangle^b \cdot \nabla\right)\right] - \mathbf{u}^{bb}_{mac}\cdot\nabla - k^{bb}_{mac}\Delta + G^b. \tag{2.98}$$

This is a dual-phase-lagging heat conduction equation with τ_q and τ_T as the phase lags of the heat flux and the temperature gradient, respectively.[16,26,48] Here, $F(\mathbf{r},t)$ is the volumetric heat source. k_e, ρc, and α are the effective thermal conductivity, volumetric heat capacity, and diffusivity, respectively. They depend not only on the thermal and physical properties of the two phases but also on the microstructure in biological tissues. Although the heat conduction in blood and tissue is assumed to be Fourier-type at the microscale [Eqs. (2.3) and (2.4)], it is a DPL-type at the macroscale.

It is interesting to note that the nontraditional convection terms $-\mathbf{u}^{bb}\cdot\nabla\left\langle T^b_{mic}\right\rangle^b - \mathbf{u}^{bt}\cdot\nabla\left\langle T^t_{mic}\right\rangle^t$ and $-\mathbf{u}^{tb}\cdot\nabla\left\langle T^b_{mic}\right\rangle^b$ $-\mathbf{u}^{tt}\cdot\nabla\left\langle T^t_{mic}\right\rangle^t$ in Eqs. (2.72) and (2.77) do lead to the appearance of the nontraditional convective terms $\mathbf{u}^{bb}_{mac}\cdot\nabla\left\langle T^i_{mic}\right\rangle^i_i$, $\mathbf{u}^{tb}_{mac}\cdot\nabla\left\langle T^i_{mic}\right\rangle^i$, $\mathbf{u}^{bt}_{mac}\cdot\nabla\left\langle T^i_{mic}\right\rangle^i$, and $\mathbf{u}^{tt}_{mac}\cdot\nabla\left\langle T^i_{mic}\right\rangle^i$ in Eqs. (2.88) and (2.91). The velocitylike terms also appear in the source terms of Eqs. (2.88) and (2.91). Furthermore, the heat source $\varepsilon^t_{mac}\left\langle \Phi^t_{mic}\right\rangle^t$ (which may come from the metabolic reaction in the tissue or external heat supply) and the convective term $\left\langle \mathbf{v}^b_{mic}\right\rangle^b\cdot\nabla\left\langle T^i_{mic}\right\rangle^i$ appear in both energy equations [Eqs. (2.88) and (2.91)]. Therefore, they are with their macroscale manifestation in both blood and tissue. The blood-tissue interaction generates a very rich way that the blood-tissue interfacial convective heat transfer, the blood velocity, the blood perfusion and the thermal source in tissue affect $\left\langle T^b_{mic}\right\rangle^b$ and $\left\langle T^t_{mic}\right\rangle^t$ [Eqs. (2.91) and (2.96)]. It would be very difficult to model these rich interactions by the mixture theory of continuum mechanics.

Consider

$$\frac{\tau_T}{\tau_q} = 1 + \frac{\left(\gamma^b_{mac}\right)^2 G^t k^{tt}_{mac} + \left(\gamma^t_{mac}\right)^2 G^b k^{bb}_{mac} - \gamma^b_{mac}\gamma^t_{mac}\left(G^b k^{tb}_{mac} + G^t k^{bt}_{mac}\right)}{\gamma^b_{mac}\gamma^t_{mac}k_e}. \tag{2.99}$$

It could be larger, equal, or smaller than 1 depending on the sign of $\left(\gamma^b_{mac}\right)^2 G^t k^{tt}_{mac} + \left(\gamma^t_{mac}\right)^2 G^b k^{bb}_{mac} - \gamma^b_{mac}\gamma^t_{mac}\left(G^b k^{tb}_{mac} + G^t k^{bt}_{mac}\right)$. By the condition for the existence of thermal waves that requires $\tau_T/\tau_q < 1$,[16,57] we may have thermal waves in bioheat transport when

$$\begin{aligned}&\left(\gamma^b_{mac}\right)^2 G^t k^{tt}_{mac} + \left(\gamma^t_{mac}\right)^2 G^b k^{bb}_{mac} - \gamma^b_{mac}\gamma^t_{mac}\left(G^b k^{tb}_{mac} + G^t k^{bt}_{mac}\right)\\ &\quad= \left(\gamma^b_{mac}\sqrt{G^t k^{tt}_{mac}} - \gamma^t_{mac}\sqrt{G^b k^{bb}_{mac}}\right)^2\\ &\qquad+ \gamma^b_{mac}\gamma^t_{mac}\left(2\sqrt{G^b G^t k^{bb}_{mac}k^{tt}_{mac}} - G^b k^{tb}_{mac} - G^t k^{bt}_{mac}\right) < 0.\end{aligned} \tag{2.100}$$

A necessary (but not sufficient) condition for Eq. (2.100) is $G^b k^{tb}_{mac} + G^t k^{bt}_{mac} > 2\sqrt{G^b G^t k^{bb}_{mac}k^{tt}_{mac}}$. When the coupling thermal conductivity term k^{bt}_{mac} and k^{tb}_{mac} are excluded so that τ_T/τ_q is always larger than 1, thermal waves would not appear. Moreover, there is a time-dependent source term $F(\mathbf{r},t)$ in the DPL macroscale bioheat equations [Eqs. (2.88) and (2.91)]. Therefore, the resonance can also occur. Note also that τ_T, τ_q and the ratio τ_T/τ_q are all G^b- and G^t- dependent. This phenomenon is peculiar to bio-tissues because of the blood perfusion process.

The rigorously-developed and closed macroscale bioheat model shows: (i) the DPL bioheat transport at macroscale for both blood and tissue phases, and (ii) the sophisticated effects of the interfacial convective heat transfer, the blood velocity, the perfusion and the metabolic heat generation on macroscale temperature fields in blood and tissues. Specially, the blood perfusion leads to a thermal source in the flux boundary condition and a surface integral term in the governing equation for blood temperature field. The resulted macroscale model has thus some distinctive features from that for impermeable-interface-systems,[58,59] such as the appearance of the nontraditional convective terms in the uncoupled DPL equations for both blood and tissue phases.

The DPL heat transport differs from the classical Fourier heat transport mainly on its existence of thermal waves and possible resonance. Such waves and resonance come from the blood-tissue coupled conduction and will vary features of heat transport significantly. In the next part, we present a general methodology for solving the mixed problems and the Cauchy problems of DPL heat-conduction equations.

III. SOLUTION FOR DPL BIOHEAT EQUATIONS

We investigate the solutions of mixed initial-boundary value problems and Cauchy problems of DPL bioheat equations in Secs. III A and III B, respectively, by following the approach developed by Wang, Zhou, and Wei.[16] For mixed problems, solution structure theorems are given and proved for Cartesian, polar, cylindrical, and spherical coordinates in Sec. III A 1 (Theorems 1, 2 and 1′), Sec. III A 2 (Theorems 3 and 4), Sec. III A 3 (Theorems 5 and 6), and Sec. III A 4 (Theorems 7 and 8), respectively. The method of separation of variables is also illustrated in each coordinate. For Cauchy problems, solution structure Theorems 9 and 10 are given and proved, with a brief discussion on the integral transformation and perturbation method.

A. Solution structure theorems for mixed problems of DPL equations

The temperature field of mixed initial-boundary value problems in DPL heat conduction comes from the contributions of the initial temperature distribution, initial rate of temperature change, boundary temperature distribution and source term. Since the effect of the boundary distribution can be transformed to the effect of source term through the homogenization of boundary conditions, we only study the solution structure of problems under linear homogeneous boundary conditions

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T+f(M,t); \;\; \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}=0; (0,+\infty) \\ T(M,0)=\varphi(M), \quad \frac{\partial T}{\partial t}(M,0)=\psi(M); \quad \Omega, \end{cases} \tag{3.1}$$

where τ_0, A, and B are constants. $f(M,t)$, $\varphi(M)$, and $\psi(M)$ are known functions. Δ is the Laplacian. $\mathrm{L}(T,\partial T/\partial n)|_{\partial\Omega} = 0$ denotes the linear homogeneous boundary conditions of all three types (Dirichlet, Neumann, and Robin boundary conditions), with $\partial T/\partial n$ standing for the normal derivative on the boundary $\partial\Omega$. For one-, two-, and three-dimensional problems, there are 9, 81, and 729 combinations of linear homogeneous boundary conditions, respectively. When $T=T(M,t)$ stands for the temperature at spatial point M (in the space domain Ω) and time instant t, the equation describes a DPL heat conduction with $\tau_0=\tau_q$, $A^2=\alpha/\tau_q$, and $B^2=\alpha\tau_T/\tau_q$.

According to the principle of superposition, the solution of problem (3.1) is the summation of solutions of the following three problems:

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}=0; (0,+\infty) \\ T(M,0)=0, \quad \frac{\partial T}{\partial t}(M,0)=\psi(M); \quad \Omega, \end{cases} \tag{3.2}$$

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}=0; (0,+\infty) \\ T(M,0)=\varphi(M), \quad \frac{\partial T}{\partial t}(M,0)=0; \quad \Omega, \end{cases} \tag{3.3}$$

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T+f(M,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}=0; (0,+\infty) \\ T(M,0)=0, \quad \frac{\partial T}{\partial t}(M,0)=0; \quad \Omega. \end{cases} \tag{3.4}$$

The *solution structure theorems* relate the solution of (3.2) with those of (3.3) and (3.4), and thus considerably simplify the development of solutions of DPL heat conduction equations by only solving the problem regarding the ψ-contribution. In the following, we first discuss the solution of the ψ-contribution problem (3.2) for the sake of completeness, and then give and prove the solution structure theorems for Cartesian, polar, cylindrical and spherical coordinates, respectively.

1. Cartesian coordinates

We find the solution of (3.2) by using the method of separation of variables, for one-, two-, and three-dimensional cases successively.

For *one-dimensional case*, the problem (3.2) becomes

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\frac{\partial^2 T}{\partial x^2}+B^2\frac{\partial^3 T}{\partial t\partial x^2}; \quad (0,l_1)\times(0,+\infty) \\ -b_1\frac{\partial T}{\partial x}(0,t)+k_1T(0,t)=0, b_2\frac{\partial T}{\partial x}(l_1,t)+k_2T(l_1,t)=0 \\ T(x,0)=0, \quad \frac{\partial T}{\partial t}(x,0)=\psi(x), \end{cases} \tag{3.5}$$

where b_i and k_i $(i=1,2)$ are non-negative real constants and $b_i+k_i>0 (i=1,2)$. Consider the nontrivial solution of (3.5) with the form of

$$T(x,t)=\Gamma(t)X(x),$$

where $\Gamma(t)$ and $X(x)$ are functions to be determined of the only variables present. Substituting this into the Eq. (3.5) yields

$$\frac{1}{\tau_0}\Gamma'(t)X(x)+\Gamma''(t)X(x)=A^2\Gamma(t)X''(x)+B^2\Gamma'(t)X''(x),$$

where the single and double primes ($'$ and $''$) denote the first- and second-order derivatives, respectively, with respect to the only variable present. By separation of variables, we have

$$\frac{\frac{1}{\tau_0}\Gamma'(t)+\Gamma''(t)}{A^2\Gamma(t)+B^2\Gamma'(t)}=\frac{X''(x)}{X(x)}=-\lambda,$$

where $-\lambda$ is the separation constant. The separation equation for the temporal part $\Gamma(t)$ is thus

$$\Gamma''(t)+\left(\frac{1}{\tau_0}+\lambda B^2\right)\Gamma'(t)+\lambda A^2\Gamma(t)=0, \tag{3.6}$$

and the homogeneous system for the spatial part $X(x)$ is

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ -b_1X'(0)+k_1X(0)=0, b_2X'(l_1)+k_2X(l_1)=0 \end{cases}. \tag{3.7}$$

The problem (3.7) is called an *eigenvalue problem* because it has solutions only for certain values of the separation constant $\lambda=\lambda_n (n=1,2,3,...)$, which are called the *eigenvalues*. The corresponding solutions $X_n(x)$ are called the *eigenfunctions* of the problem.

The eigenvalue problem (3.7) is a general form encompassing nine problems corresponding to nine combinations of boundary conditions. As an example to illustrate the determination of λ and $X(x)$, we consider the case of all nonzero b_1, k_1, b_2, and k_2

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)-h_1X(0)=0, X'(l_1)+h_2X(l_1)=0 \\ h_1=k_1/b_1, h_2=k_2/b_2. \end{cases} \tag{3.8}$$

If $\lambda = 0$, its general solution is $X(x) = c_1 x + c_2$. Applying the boundary conditions yields

$$\begin{cases} c_1 - h_1 c_2 = 0 \\ c_1(1 + h_2 l_1) + h_2 c_2 = 0. \end{cases}$$

Its solution is $c_1 = c_2 = 0$, since $\Delta = \begin{vmatrix} 1 & -h_1 \\ 1 + h_2 l_1 & h_2 \end{vmatrix} \neq 0$. We thus obtain the trivial solution $X(x) \equiv 0$. Therefore, λ cannot be zero.

If $\lambda < 0$, the general solution of the equation in (3.8) reads

$$X(x) = c_1 \exp(-ax) + c_2 \exp(ax),$$

where $a^2 = -\lambda$ and $a > 0$. Applying the boundary conditions leads to

$$\begin{cases} -ac_1 + ac_2 = 0 \\ -a\exp(-al_1) + a\exp(al_1) = 0. \end{cases}$$

Its solution is still $c_1 = c_2 = 0$. We thus again obtain the trivial solution $X(x) \equiv 0$. Therefore, the eigenvalues of (3.8) must be positive.

For positive $\lambda = b^2 > 0$, the general solution of the equation in (3.8) reads

$$X(x) = c_1 \cos bx + c_2 \sin bx.$$

Applying the boundary conditions yields

$$\begin{cases} bc_2 - h_1 c_1 = 0, \textit{ or } c_2 = h_1 c_1 / b \\ -c_1 b \sin bl_1 + c_2 b \cos bl_1 + h_2 c_1 \cos bl_1 + h_2 c_2 \sin bl_1 = 0. \end{cases}$$

Either c_1 or c_2 cannot be zero to have a nontrivial solution. Its solution is, by noting that h_1 and h_2 are physically positive values,

$$\cot bl_1 = \frac{1}{h_1 + h_2}\left(b - \frac{h_1 h_2}{b}\right) = \frac{1}{(h_1 + h_2) l_1}\left(bl_1 - \frac{h_1 h_2 l_1^2}{bl_1}\right).$$

Let

$$f(x) = \cot x - \frac{1}{(h_1 + h_2) l_1}\left(x - \frac{h_1 h_2 l_1^2}{x}\right). \tag{3.9}$$

bl_1 thus represents the zero points of $f(x)$. Since $f(x)$ is an odd function and $\lambda = b^2$, we wish to find the positive zero points of $f(x)$ only. Letting μ_n be the n-th positive zero point of $f(x)$, we have eigenvalues

$$\lambda_n = b_n^2 = \left(\frac{\mu_n}{l_1}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions can be written as

$$\begin{aligned} X_n(x) &= \frac{\mu_n}{l_1 h_1}\cos\frac{\mu_n x}{l_1} + \sin\frac{\mu_n x}{l_1} \\ &= \sqrt{1 + \left(\frac{\mu_n}{l_1 h_1}\right)^2}\sin\left(\frac{\mu_n x}{l_1} + \phi_n\right), \end{aligned} \tag{3.10}$$

where $\tan\phi_n = \frac{\mu_n}{l_1 h_1}$.

To summarize, we have eigenvalues $\lambda_n = (\mu_n / l_1)^2$, with μ_n being the positive zero points of $f(x)$ in Eq. (3.9); eigenfunctions $X_n(x) = \sin(\mu_n x / l_1 + \phi_n)$, with $\tan\phi_n = \mu_n / (l_1 h_1)$. Normal square of eigenfunction set is

$$\begin{aligned} \|X_n(x)\|^2 &= \int_0^{l_1}\sin^2\left(\frac{\mu_n x}{l_1} + \phi_n\right)dx \\ &= \frac{l_1}{\mu_n}\int_0^{l_1}\sin^2\left(\frac{\mu_n x}{l_1} + \phi_n\right)d\left(\frac{\mu_n x}{l_1} + \phi_n\right) \\ &= \frac{l_1}{\mu_n}\int_0^{l_1}\frac{1}{2}\left[1 - \cos 2\left(\frac{\mu_n x}{l_1} + \phi_n\right)\right]d\left(\frac{\mu_n x}{l_1} + \phi_n\right) \\ &= \frac{l_1}{\mu_n}\left[\frac{1}{2}\left(\frac{\mu_n x}{l_1} + \phi_n\right) - \frac{1}{4}\sin 2\left(\frac{\mu_n x}{l_1} + \phi_n\right)\right]\Bigg|_0^{l_1} \\ &= \frac{l_1}{2}\left[1 - \frac{\sin\mu_n}{\mu_n}\cos(\mu_n + 2\phi_n)\right]. \end{aligned} \tag{3.11}$$

Since the two characteristic roots of (3.6) are

$$\begin{aligned} r_{1,2} &= \frac{1}{2}\left[-\left(\frac{1}{\tau_0} + \lambda_n B^2\right) \pm \sqrt{\left(\frac{1}{\tau_0} + \lambda_n B^2\right)^2 - 4\lambda_n A^2}\right] \\ &= \alpha_n \pm \beta_n \mathbf{i}, \end{aligned}$$

where

$$\begin{cases} \alpha_n = -\frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_n B^2\right) \\ \beta_n = \frac{1}{2}\sqrt{4\lambda_n A^2 - \left(\frac{1}{\tau_0} + \lambda_n B^2\right)^2}. \end{cases} \tag{3.12}$$

The solution of (3.6) reads

$$\Gamma_{\mathrm{n}}(t) = e^{\alpha_n t}(a_n \cos\beta_n t + b_n \underline{\sin}\beta_n t).$$

Thus the solution of (3.5) has the form of

$$T(x,t) = \sum_{n=1}^{\infty} e^{\alpha_n t}(a_n \cos\beta_n t + b_n \underline{\sin}\beta_n t)\sin\left(\frac{\mu_n x}{l_1} + \phi_n\right), \tag{3.13}$$

where a_n and b_n are both constants to be determined. $\underline{\sin}\beta_n t$ is defined by

$$\underline{\sin}\beta_n t = \begin{cases} \sin\beta_n t, \beta_n \neq 0 \\ t, \beta_n = 0. \end{cases}$$

Applying the initial condition $T(x,0) = 0$ yields $a_n = 0$. Applying the initial condition $\partial T / \partial t(x,0) = \psi(x)$, b_n can be determined by

$$\sum_{n=1}^{\infty} b_n \beta_n \sin\left(\frac{\mu_n x}{l_1} + \phi_n\right) = \psi(x),$$

so that

$$b_n = \frac{1}{N_n \underline{\beta}_n}\int_0^{l_1}\psi(x)\sin\left(\frac{\mu_n x}{l_1} + \phi_n\right)dx,$$

TABLE II. Eigenvalues, eigenfunctions, and normal square of eigenfunctions for nine combinations of boundary conditions.

$x=0$	$x=l$	Eigenvalues	Eigenfunctions	Normal square M_m	Notes
1)	$X=0$	$\left(\frac{m\pi}{l}\right)^2$	$\sin\frac{m\pi x}{l}$	$\frac{l}{2}$	$m=1,2,\ldots$
2) $X=0$	$X'=0$	$\left[\frac{(2m+1)\pi}{2l}\right]^2$	$\sin\frac{(2m+1)\pi x}{2l}$	$\frac{l}{2}$	$m=0,1,\ldots$
3)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\frac{\mu_m x}{l}$	$\frac{l}{2}\left(1-\frac{\sin 2\mu_m}{2\mu_m}\right)$	$\mu_m(m=1,2,\ldots)$ are positive zero-points of $f(x)=\tan x+\frac{x}{lh_2}$
4)	$X=0$	$\left[\frac{(2m+1)\pi}{2l}\right]^2$	$\cos\frac{(2m+1)\pi x}{2l}$	$\frac{l}{2}$	$m=0,1,\ldots$
5) $X'=0$	$X'=0$	$\left(\frac{m\pi}{l}\right)^2$	$\cos\frac{m\pi x}{l}$	$l,\ \frac{l}{2}$	$m=0,1,\ldots$
6)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\cos\frac{\mu_m x}{l}$	$\frac{l}{2}\left(1+\frac{\sin 2\mu_m}{2\mu_m}\right)$	$\mu_m(m=1,2,\ldots)$ are positive zero-points of $g(x)=\cot x-\frac{x}{lh_2}$
7)	$X=0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\frac{\mu_m}{lh_1}$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\ldots)$ are positive zero-points of $f(x)=\tan x+\frac{x}{lh_1}$
8) $X'-h_1X=0$ $h_1>0$	$X'=0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\frac{\mu_m}{lh_1}$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\ldots)$ are positive zero-points of $f(x)=\cot x-\frac{x}{lh_1}$
9)	$X'+h_2X=0$ $h_2>0$	$\left(\frac{\mu_m}{l}\right)^2$	$\sin\left(\frac{\mu_m}{l}x+\varphi_m\right)$ $\tan\varphi_m=\frac{\mu_m}{lh_1}$	$\frac{l}{2}\left[1-\frac{\sin\mu_m}{\mu_m}\cdot\cos(\mu_m+2\varphi_m)\right]$	$\mu_m(m=1,2,\ldots)$ are positive zero-points of $f(x)=\cot x-\frac{1}{l(h_1+h_2)}\cdot\left(x-\frac{l^2h_1h_2}{x}\right)$

where $\underline{\beta}_n$ is defined by

$$\underline{\beta}_n=\begin{cases}\beta_n, \beta_n\neq 0\\ 1, \beta_n=0.\end{cases}$$

N_n is the normal square of $\left\{\sin\left(\frac{\mu_n x}{l_1}+\phi_n\right)\right\}$, given by Eq. (3.11).

Finally, we have the solution of (3.5)

$$\begin{cases}T(x,t)=\displaystyle\sum_{n=1}^{\infty}b_n e^{\alpha_n t}\underline{\sin}\beta_n t\sin\left(\frac{\mu_n x}{l_1}+\phi_n\right)\\ b_n=\displaystyle\frac{1}{N_n\underline{\beta}_n}\int_0^{l_1}\psi(x)\sin\left(\frac{\mu_n x}{l_1}+\phi_n\right)dx.\end{cases}\tag{3.14}$$

λ_n, $X_n(x)$, and N_n of the other eight combinations of boundary conditions can also be obtained using a similar approach. The results for all nine combinations are listed in Table II.

$T(x,t)=W_\psi(x,t)$ given by (3.14) actually enjoys a very elegant structure. We may use this structure and Table II to write out $W_\psi(x,t)$ directly. Let λ_n, $X_n(x)$, and N_n be the eigenvalues, eigenfunctions, and normal square of eigenfunctions from Table II based on given boundary conditions. The structure of $W_\psi(x,t)$ is thus

$$\begin{cases}W_\psi(x,t)=\displaystyle\sum_n b_n e^{\alpha_n t}\underline{\sin}\beta_n t\cdot X_n(x)\\ b_n=\displaystyle\frac{1}{N_n\underline{\beta}_n}\int_0^{l_1}\psi(x)X_n(x)dx,\end{cases}\tag{3.15}$$

where $\sum_n$ denotes either $\sum_{n=1}^{\infty}$ or $\sum_{n=0}^{\infty}$ depending on the boundary conditions, α_n and β_n are given by Eq. (3.12). If $\Delta=(1/\tau_0+\lambda_n B^2)^2-4\lambda_n A^2>0$ so that β_n is purely imaginary for some n, we can change $\sin\beta_n t$ into $\left(e^{\mathbf{i}\beta_n t}-e^{-\mathbf{i}\beta_n t}\right)/(2\mathbf{i})$. The general term of the series solution decays very quickly toward zero for all cases of $\Delta>0$, $=0$, and <0, which facilitates its applications of taking only the first few terms.

Some important properties of eigenvalue problems include, for example: (i) all eigenvalues are non-negative and real-valued for all combinations of boundary conditions; a vanished eigenvalue appears only when $X'(0)=X'(l_1)=0$; (ii) eigenvalues form a sequence of numbers which monotonically increases toward infinity, whatever the boundary conditions, i.e., $0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots$, $\lim_{k\to\infty}\lambda_k=\infty$; (iii) all eigenfunction sets $\{X_n(x)\}$ are orthogonal in $[0,\ l_1]$, i.e., $(X_n,X_m)=\int_0^{l_1}X_n(x)X_m(x)dx=0$, $n\neq m$; (iv) any function $f(x)\in L^2[a,b]$ can be expanded into a generalized Fourier series by an eigenfunction set, i.e.,

$$\begin{cases}f(x)=\displaystyle\sum_{n=1}^{\infty}c_nX_n(x)\\ c_n=\displaystyle\frac{1}{N_n}\int_a^b X_n(x)f(x)dx, N_n=\int_a^b X_n^2(x)dx,\end{cases}$$

where $\sqrt{N_n}$ is called the normal of $\{X_n(x)\}$ and serves as the measure of function size. Therefore, $\{X_n(x)\}$ forms a complete and orthogonal set in $[a, b]$ and $\lim_{M_0\to\infty}\sqrt{\int_a^b \left[f(x)-\sum_{n=1}^{M_0} c_n X_n(x)\right]^2 dx} = 0$.

Similarly for *two-dimensional* case, problem (3.2) becomes

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}\right)+B^2\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}\right); \mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial n}\right)\Big|_{\partial \mathrm{D}}=0 \\ T(x,y,0)=0, \quad \dfrac{\partial T}{\partial t}(x,y,0)=\psi(x,y), \end{cases} \tag{3.16}$$

where D is the rectangular domain:$0<x<l_1, 0<y<l_2$; ∂D denotes its boundary. Its solution $W_\psi(x,y,t)$ has the form of

$$\begin{cases} W_\psi(x,y,t)=\sum_{n,m} b_{nm}e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t\cdot X_n(x)Y_m(y) \\ b_{nm}=\dfrac{1}{N_nN_m\underline{\beta}_{nm}}\iint\limits_{\mathrm{D}}\psi(x,y)X_n(x)Y_m(y)dxdy, \end{cases} \tag{3.17}$$

where $\sum_{n,m}$ is a double summation and denotes $\sum_{n=1,m=1}^{\infty}$, $\sum_{n=0,m=0}^{\infty}$, $\sum_{n=1,m=0}^{\infty}$, or $\sum_{n=0,m=0}^{\infty}$ depending on the given boundary conditions in the x- and y-directions. α_{nm} and β_{nm} are

$$\alpha_{nm}=-\frac{1}{2}\left[\frac{1}{\tau_0}+(\lambda_n+\lambda_m)B^2\right],$$

$$\beta_{nm}=\frac{1}{2}\sqrt{4(\lambda_n+\lambda_m)A^2-\left[\frac{1}{\tau_0}+(\lambda_n+\lambda_m)B^2\right]^2}.$$

λ_n, $X_n(x)$, and N_n are the eigenvalues, eigenfunctions, and normal square of eigenfunctions, respectively, in the x-direction. λ_m, $Y_m(y)$, and N_m are the eigenvalues, eigenfunctions, and normal square of eigenfunctions, respectively, in the y-direction. They can be obtained directly from Table II based on the corresponding boundary conditions.

For *three-dimensional cases*, problem (3.2) becomes

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\dfrac{\partial^2 T}{\partial z^2}\right)+B^2\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}+\dfrac{\partial^2 T}{\partial z^2}\right); \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial n}\right)\Big|_{\partial \Omega}=0 \\ T(x,y,z,0)=0, \quad \dfrac{\partial T}{\partial t}(x,y,z,0)=\psi(x,y,z), \end{cases} \tag{3.18}$$

where Ω is the cubic domain: $0<x<l_1, 0<y<l_2, 0<z<l_3$. $\partial\Omega$ denotes its boundary. Its solution $W_\psi(x,y,z,t)$ reads

$$\begin{cases} W_\psi(x,y,z,t)=\sum_{n,m,k} b_{nmk}e^{\alpha_{nmk}t}\underline{\sin}\beta_{nmk}t\cdot X_n(x)Y_m(y)Z_k(z) \\ b_{nmk}=\dfrac{1}{N_nN_mN_k\underline{\beta}_{nmk}}\iiint\limits_{\Omega}\psi(x,y,z)X_n(x)Y_m(y)Z_k(z)dxdydz, \end{cases} \tag{3.19}$$

where $\sum_{n,m,k}$ is a triple summation and has eight possibilities depending on the given boundary conditions in the x-, y-, and z-directions. α_{nmk} and β_{nmk} are given by

$$\alpha_{nmk}=-\frac{1}{2}\left[\frac{1}{\tau_0}+(\lambda_n+\lambda_m+\lambda_k)B^2\right],$$

$$\beta_{nmk}=\frac{1}{2}\sqrt{4(\lambda_n+\lambda_m+\lambda_k)A^2-\left[\frac{1}{\tau_0}+(\lambda_n+\lambda_m+\lambda_k)B^2\right]^2}.$$

λ_n, λ_m, λ_k, $X_n(x)$, $Y_m(y)$, $Z_k(z)$, N_n, N_m, and N_k are the eigenvalues, eigenfunctions, and normal squares of eigenfunctions in the three directions, respectively. They can be obtained from Table II based on the corresponding boundary conditions. Note that we can also use other methods, such as Fourier expansion, to obtain the same results.

Now we give Theorems 1 and 2 which relate the solution of the $\psi-$ contribution problem (3.2) with the solutions of φ − contribution problem (3.3) and $f-$ contribution problem (3.4), respectively, for Cartesian coordinates.

Theorem 1. Let $W_\psi(M,t)$ denote the solution of (3.2). The solution of (3.3) can be written as

$$T(M,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)+B^2W_{\lambda\varphi}(M,t), \tag{3.20}$$

where

$$\lambda=\begin{cases} \lambda_n, & \text{one-dimensional (1D) case} \\ \lambda_n+\lambda_m, & \text{two-dimensional (2D) case} \\ \lambda_n+\lambda_m+\lambda_k, & \text{three-dimensional (3D) case,} \end{cases}$$

λ_n, λ_m, and λ_k are the eigenvalues corresponding to the eigenvalue problems in the x-, y- and z-directions, respectively.

Proof. Again starting from *one-dimensional cases*, we need to prove that the solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2 \dfrac{\partial^2 T}{\partial x^2} + B^2 \dfrac{\partial^3 T}{\partial t \partial x^2}; \quad (0, l_1) \times (0, +\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial n}\right)\Big|_{x=0,l_1} = 0 \\ T(x,0) = \varphi(x), \quad \dfrac{\partial T}{\partial t}(x,0) = 0 \end{cases} \tag{3.21}$$

is

$$T(x,t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(x,t) + B^2 W_{\lambda_n \varphi}(x,t), \tag{3.22}$$

where W_φ and $W_{\lambda_n\varphi}$ are both with the form of Eq. (3.15) but replacing $\psi(x)$ with $\varphi(x)$ and $\lambda_n\varphi(x)$, respectively, which has been obtained by using the method of separation of variables. By the same method, problem (3.21) has the solution of

$$T(x,t) = \sum_n e^{\alpha_n t}(c_n \cos \beta_n t + d_n \underline{\sin} \beta_n t) X_n(x). \tag{3.23}$$

Thus,

$$\frac{\partial T}{\partial t}(x,t) = \sum_n e^{\alpha_n t}[\alpha_n(c_n \cos \beta_n t + d_n \underline{\sin}\beta_n t) + (-c_n\beta_n \sin \beta_n t + d_n \underline{\beta}_n \cos \beta_n t)] X_n(x).$$

Applying the two initial condition leads to

$$c_n \alpha_n + d_n \underline{\beta}_n = 0, \text{or } d_n = -\frac{\alpha_n}{\underline{\beta}_n} c_n,$$

$$c_n = \frac{1}{N_n} \int_0^{l_1} \varphi(x) X_n(x) dx,$$

where N_n is the normal square of $\{X_n(x)\}$.

We need to prove that the solutions given by (3.22) and (3.23) are the same. By Eq. (3.15), we have

$$\begin{cases} \dfrac{\partial}{\partial t} W_\varphi(x,t) = \displaystyle\sum_n b_n^* e^{\alpha_n t}\left(\alpha_n \underline{\sin}\beta_n t + \underline{\beta}_n \cos \beta_n t\right) X_n(x) \\ b_n^* = \dfrac{1}{N_n \underline{\beta}_n} \displaystyle\int_0^{l_1} \varphi(x) X_n(x) dx = \dfrac{c_n}{\underline{\beta}_n} \end{cases}$$

or

$$\begin{aligned} \frac{\partial}{\partial t} W_\varphi(x,t) &= \sum_n e^{\alpha_n t}\left(c_n \cos \beta_n t + \frac{\alpha_n}{\underline{\beta}_n} c_n \underline{\sin}\beta_n t\right) X_n(x) \\ &= \sum_n e^{\alpha_n t}(c_n \cos \beta_n t - d_n \underline{\sin}\beta_n t) X_n(x). \end{aligned}$$

Also by Eq. (3.15),

$$\begin{aligned} &\frac{1}{\tau_0} W_\varphi(x,t) + B^2 W_{\lambda_n \varphi}(x,t) \\ &= \sum_n \left(\frac{1}{\tau_0} + \lambda_n B^2\right) b_n^* e^{\alpha_n t} \underline{\sin}\beta_n t \cdot X_n(x) \\ &= \sum_n -2 \frac{\alpha_n}{\underline{\beta}_n} c_n e^{\alpha_n t} \underline{\sin}\beta_n t \cdot X_n(x) \\ &= \sum_n 2 d_n e^{\alpha_n t} \underline{\sin}\beta_n t \cdot X_n(x). \end{aligned}$$

Therefore, by adding $\partial W_\varphi(x,t)/\partial t$ and $(1/\tau_0)W_\varphi(x,t) + B^2 W_{\lambda_n\varphi}(x,t)$, the solution given by (3.22) is indeed equal to that given by (3.23).

Similarly, for *two-dimensional cases*, the solution of the $\varphi(x,y)$ – contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right) + B^2 \dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right); \mathrm{D} \times (0, +\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial n}\right)\Big|_{\partial \mathrm{D}} = 0 \\ T(x,y,0) = \varphi(x,y), \quad \dfrac{\partial T}{\partial t}(x,y,0) = 0 \end{cases} \tag{3.24}$$

can be obtained by separation of variables as

$$\begin{cases} T(x,y,t) = \displaystyle\sum_{n,m} e^{\alpha_{nm} t}(c_{nm} \cos \beta_{nm} t + d_{nm} \underline{\sin}\beta_{nm} t) X_n(x) Y_m(y) \\ c_{nm} = \dfrac{1}{N_n N_m} \displaystyle\iint_{\mathrm{D}} \varphi(x,y) X_n(x) Y_m(y) dxdy \\ d_{nm} = -\dfrac{\alpha_{nm}}{\underline{\beta}_{nm}} c_{nm}, \end{cases} \tag{3.25}$$

where N_n and N_m are the normal squares of $\{X_n(x)\}$ and $\{Y_m(y)\}$.

By Eq. (3.17), we have

$$\begin{cases} \dfrac{\partial}{\partial t}W_\varphi(x,y,t) = \sum_{n,m} b^*_{nm} e^{\alpha_{nm}t}\left(\alpha_{nm}\underline{\sin}\beta_{nm}t + \underline{\beta}_{nm}\cos\beta_{nm}t\right)X_n(x)Y_m(y) \\ b^*_{nm} = \dfrac{1}{N_nN_m\underline{\beta}_{nm}}\iint\limits_{D}\varphi(x,y)X_n(x)Y_m(y)dxdy = \dfrac{c_{nm}}{\underline{\beta}_{nm}} \end{cases}$$

or

$$\frac{\partial}{\partial t}W_\varphi(x,y,t) = \sum_{n,m} e^{\alpha_{nm}t}\left(c_{nm}\cos\beta_{nm}t + \frac{\alpha_{nm}}{\underline{\beta}_m}c_{nm}\underline{\sin}\beta_{nm}t\right)X_n(x)Y_m(y) = \sum_{n,m} e^{\alpha_{nm}t}(c_{nm}\cos\beta_{nm}t - d_{nm}\underline{\sin}\beta_{nm}t)X_n(x)Y_m(y).$$

Also,

$$\begin{aligned} \frac{1}{\tau_0}W_\varphi(x,y,t) + B^2W_{(\lambda_n+\lambda_m)\varphi}(x,y,t) &= \sum_{n,m}\left[\frac{1}{\tau_0} + (\lambda_n+\lambda_m)B^2\right]b^*_{nm}e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t\cdot X_n(x)Y_m(y) \\ &= \sum_{n,m} -2\frac{\alpha_{nm}}{\underline{\beta}_{nm}}c_{nm}e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t\cdot X_n(x)Y_m(y) = \sum_{n,m} 2d_{nm}e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t\cdot X_n(x)Y_m(y). \end{aligned}$$

By adding $\partial W_\varphi(x,y,t)/\partial t$ and $(1/\tau_0)W_\varphi(x,y,t) + B^2\ W_{(\lambda_n+\lambda_m)\varphi}(x,y,t)$, we can prove the solution given by Theorem 1 is the same with that given by (3.25).

For *three-dimensional cases*, the $\varphi(x,y,z)$ − contribution problem becomes

We may obtain its solution directly by separation of variables

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}\right) + B^2\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}\right); \Omega\times(0,+\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial n}\right)\Big|_{\partial\Omega} = 0 \\ T(x,y,z,0) = \varphi(x,y,z), \quad \dfrac{\partial T}{\partial t}(x,y,z,0) = 0. \end{cases} \tag{3.26}$$

We may obtain its solution directly by separation of variables

$$\begin{cases} T(x,y,z,t) = \sum_{n,m,k} e^{\alpha_{nmk}t}(c_{nmk}\cos\beta_{nmk}t + d_{nmk}\underline{\sin}\beta_{nmk}t)X_n(x)Y_m(y)Z_k(z) \\ c_{nmk} = \dfrac{1}{N_nN_mN_k}\iiint\limits_{\Omega}\varphi(x,y,z)X_n(x)Y_m(y)Z_k(z)dxdydz \\ d_{nmk} = -\dfrac{\alpha_{nmk}}{\underline{\beta}_{nmk}}c_{nmk}, \end{cases} \tag{3.27}$$

where N_n, N_m, and N_k are the normal squares of $\{X_n(x)\}$, $\{Y_m(y)\}$, and $\{Z_k(z)\}$, respectively. By substituting $W_\varphi(x,y,z)$ based on Eq. (3.19) into $\partial W_\varphi/\partial t$, we have

$$\begin{cases} \dfrac{\partial}{\partial t}W_\varphi(x,y,z,t) = \sum_{n,m,k} b^*_{nmk}e^{\alpha_{nmk}t}\left(\alpha_{nmk}\underline{\sin}\beta_{nmk}t + \underline{\beta}_{nmk}\cos\beta_{nmk}t\right)X_n(x)Y_m(y)Z_k(z) \\ b^*_{nmk} = \dfrac{1}{N_nN_mN_k\underline{\beta}_{nmk}}\iiint\limits_{\Omega}\varphi(x,y,z)X_n(x)Y_m(y)Z_k(z)dxdydz = \dfrac{c_{nmk}}{\underline{\beta}_{nmk}} \end{cases}$$

or

$$\begin{aligned} \frac{\partial}{\partial t}W_\varphi(x,y,z,t) &= \sum_{n,m,k} e^{\alpha_{nmk}t}\left(c_{nmk}\cos\beta_{nmk}t + \frac{\alpha_{nmk}}{\underline{\beta}_m}c_{nmk}\underline{\sin}\beta_{nmk}t\right)X_n(x)Y_m(y)Z_k(z) \\ &= \sum_{n,m,k} e^{\alpha_{nmk}t}(c_{nmk}\cos\beta_{nmk}t - d_{nmk}\underline{\sin}\beta_{nmk}t)X_n(x)Y_m(y)Z_k(z) \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\tau_0}W_\varphi(x,y,z,t)+B^2W_{(\lambda_n+\lambda_m+\lambda_k)\varphi}(x,y,z,t)\\
&=\sum_{n,m,k}\left[\frac{1}{\tau_0}+(\lambda_n+\lambda_m+\lambda_k)B^2\right]b^*_{nmk}e^{\alpha_{nmk}t}\underline{\sin}\beta_{nmk}t\\
&\cdot X_n(x)Y_m(y)Z_k(z)\\
&=\sum_{n,m,k}-2\frac{\alpha_{nmk}}{\underline{\beta}_{nmk}}c_{nmk}e^{\alpha_{nmk}t}\underline{\sin}\beta_{nmk}t\cdot X_n(x)Y_m(y)Z_k(z)\\
&=\sum_{n,m,k}2d_{nmk}e^{\alpha_{nmk}t}\underline{\sin}\beta_{nmk}t\cdot X_n(x)Y_m(y)Z_k(z).
\end{aligned}$$

It is thus easy to prove that $(1/\tau_0+\partial/\partial t)W_\varphi(x,y,z,t)$ $+B^2W_{(\lambda_n+\lambda_m+\lambda_k)\varphi}(x,y,z,t)$ gives the same solution as that in (3.27). Therefore, we have successively proved Theorem 1 for 1D, 2D, and 3D cases in Cartesian coordinates.

Theorem 2

Let $W_\psi(M,t)$ denote the solution of (3.2). The solution of (3.4) can be written as

$$T(M,t)=\int_0^t W_{f_\tau(M,t-\tau)}d\tau, \tag{3.28}$$

where $f_\tau=f(M,\tau)$.

Proof. We need to prove that the solution given by Eq. (3.28) satisfies the equation, the boundary conditions and the initial conditions in problem (3.4). By the definition of $W_{f_\tau}(M,t-\tau)$, we have

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t}+\dfrac{\partial^2 W_{f_\tau}}{\partial t^2}-A^2\Delta W_{f_\tau}-B^2\dfrac{\partial}{\partial t}\Delta W_{f_\tau}=0\\ \mathrm{L}\left(W_{f_\tau},\dfrac{\partial W_{f_\tau}}{\partial n}\right)\Big|_{\partial\Omega}=0\\ W_{f_\tau}(M,t-\tau)\big|_{t=\tau}=0,\quad \dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\Big|_{t=\tau}=f(M,\tau).\end{cases} \tag{3.29}$$

After substituting Eq. (3.28) into the equation of (3.4) and applying the equation in (3.29), we obtain

$$\begin{aligned}
\frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}-A^2\Delta T-B^2\frac{\partial}{\partial t}\Delta T&=\frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)d\tau+\frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}(M,t-\tau)d\tau\\
&\quad-A^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)d\tau-B^2\frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}(M,t-\tau)d\tau\\
&=\frac{1}{\tau_0}\left[\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}d\tau+W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]+\int_0^t\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}d\tau+\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\Big|_{\tau=t}\\
&\quad-A^2\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau-B^2\frac{\partial}{\partial t}\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau\\
&=\frac{1}{\tau_0}\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}d\tau+\int_0^t\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}d\tau+f(M,t)\\
&\quad-A^2\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau-B^2\left[\int_0^t\frac{\partial}{\partial t}\Delta W_{f_\tau}(M,t-\tau)d\tau+\Delta W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]\\
&=\int_0^t\left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}+\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}-A^2\Delta W_{f_\tau}(M,t-\tau)-B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}(M,t-\tau)\right)d\tau\\
&\quad+f(M,t)=f(M,t)
\end{aligned}$$

in which $W_{f_\tau}(M,t-\tau)\big|_{\tau=t}=0$ and $\partial W_{f_\tau}(M,t-\tau)/\partial t\big|_{\tau=t}$ $=f(M,t)$ have been used. Therefore, $T=\int_0^t W_{f_\tau}(M,t-\tau)d\tau$ given by (3.28) satisfies the equation in (3.4).

By substituting Eq. (3.28) into the boundary conditions of (3.4) and applying the boundary conditions of (3.29), we have

$$\begin{aligned}
\mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}&=\mathrm{L}\left(\int_0^t W_{f_\tau}(M,t-\tau)d\tau,\frac{\partial}{\partial n}\int_0^t W_{f_\tau}(M,t-\tau)d\tau\right)\Big|_{\partial\Omega}\\
&=\mathrm{L}\left(\int_0^t W_{f_\tau}(M,t-\tau)d\tau,\int_0^t\frac{\partial}{\partial n}W_{f_\tau}(M,t-\tau)d\tau\right)\Big|_{\partial\Omega}\\
&=\int_0^t\mathrm{L}\left(W_{f_\tau}(M,t-\tau),\frac{\partial}{\partial n}W_{f_\tau}(M,t-\tau)\right)\Big|_{\partial\Omega}d\tau=0.
\end{aligned}$$

Therefore, $T=\int_0^t W_{f_\tau}(M,t-\tau)d\tau$ given by (3.28) satisfies the boundary conditions in (3.4). Also,

$$\int_0^t W_{f_\tau}(M,t-\tau)d\tau\Big|_{t=0}=0,$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)d\tau\Big|_{t=0}\\
&\quad=\left[\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}d\tau+W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]\Big|_{t=0}=0.
\end{aligned}$$

Equation (3.28) also satisfies the initial conditions in (3.4). We have thus successfully proved Theorem 2 for Cartesian coordinates.

As a summary, the solution of

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T+f(M,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\frac{\partial T}{\partial n}\right)\Big|_{\partial\Omega}=0;(0,+\infty) \\ T(M,0)=\varphi(M), \quad \frac{\partial T}{\partial t}(M,0)=\psi(M); \quad \Omega \end{cases} \tag{3.30}$$

is

$$T(M,t)=W_\psi(M,t)+\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t) + B^2W_{\lambda\varphi}(M,t)+\int_0^t W_{f_\tau}(M,t-\tau)d\tau, \tag{3.31}$$

where Ω denotes the domain of

$$\begin{cases} 0<x<l_1, & \text{one-dimensional case} \\ 0<x<l_1, 0<y<l_2, & \text{two-dimensional case} \\ 0<x<l_1, 0<y<l_2, 0<z<l_3, & \text{three-dimensional case.} \end{cases}$$

$\partial\Omega$ is the boundary of Ω. $W_\psi(M,t)$ is the solution at $f=\varphi=0$, with the form of

$$\begin{cases} W_\psi(M,t)=\sum bF(M)e^{\alpha t}\underline{\sin}\beta t \\ b=\frac{1}{N\underline{\beta}}\int_\Omega \psi(M)F(M)d\Omega \end{cases} \tag{3.32}$$

in which $\sum$ denotes $\sum_n$, $\sum_{n,m}$, and $\sum_{n,m,k}$ for 1D, 2D, and 3D cases, respectively. n, m, and k can start from either 0 or 1 depending on the given boundary conditions in the x-, y-, or z-directions, respectively. $F(M)$, α, λ, and N are given by

$$F(M)=\begin{cases} X_n(x), & \text{one-dimensional case} \\ X_n(x)Y_m(y), & \text{two-dimensional case} \\ X_n(x)Y_m(y)Z_k(z), & \text{three-dimensional case,} \end{cases} \tag{3.33}$$

$$\alpha=-\frac{1}{2}\left(\frac{1}{\tau_0}+\lambda B^2\right), \quad \beta=\frac{1}{2}\sqrt{4\lambda A^2-\left(\frac{1}{\tau_0}+\lambda B^2\right)^2}, \tag{3.34}$$

where

$$\lambda=\begin{cases} \lambda_n, & \text{one-dimensional case} \\ \lambda_n+\lambda_m, & \text{two-dimensional case} \\ \lambda_n+\lambda_m+\lambda_k, & \text{three-dimensional case,} \end{cases} \tag{3.35}$$

$$N=\begin{cases} N_n, & \text{one-dimensional case} \\ N_nN_m, & \text{two-dimensional case} \\ N_nN_mN_k, & \text{three-dimensional case.} \end{cases} \tag{3.36}$$

The integral $\int_\Omega$ denotes the definite integral, double integral, and triple integral depending on the dimensions of Ω. The eigenvalues, eigenfunctions, and normal squares of eigenfunctions (λ_n, λ_m, λ_k, $X_n(x)$, $Y_m(y)$, $Z_k(z)$, N_n, N_m, and N_k) can be directly obtained from Table II based on the corresponding boundary conditions.

To apply the solution structure theorem and the method of separation of variables (or Fourier expansion method), the boundary conditions must be linear, homogeneous, separable, and with constant coefficients. If β is purely imaginary for some n, m, and k, we can change $\sin\beta t$ into $\left(e^{\mathbf{i}\beta t}-e^{-\mathbf{i}\beta t}\right)/(2\mathbf{i})$.

When $\varphi(M)$ and $\psi(M)$ satisfy the consistency conditions, i.e., both satisfying the boundary conditions (well-posed problem), we have another form of solution structure theorem for the $\varphi(M)$ − contribution problem:

Theorem 1′. Let $W_\psi(M,t)$ be the solution of the *well-posed* problem

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ \mathrm{L}(T,\frac{\partial T}{\partial n})\Big|_{\partial\Omega}=0;(0,+\infty) \\ T(M,0)=0, \quad \frac{\partial T}{\partial t}(M,0)=\psi(M); \quad \Omega. \end{cases} \tag{3.37}$$

The solution of the *well-posed* problem

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\frac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ \mathrm{L}(T,\frac{\partial T}{\partial n})\Big|_{\partial\Omega}=0;(0,+\infty) \\ T(M,0)=\varphi(M), \quad \frac{\partial T}{\partial t}(M,0)=0; \quad \Omega \end{cases} \tag{3.38}$$

is

$$T(M,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t)+W_{-B^2\Delta\varphi}(M,t). \tag{3.39}$$

Proof. According to Theorem 1, we only need to prove

$$B^2W_{\lambda\varphi}(M,t)=W_{-B^2\Delta\varphi}(M,t) \tag{3.40}$$

for well-posed problems. By Eq. (3.32), we have

$$\begin{cases} B^2W_{\lambda\varphi}(M,t)=\sum\lambda B^2b^*F(M)e^{\alpha t}\underline{\sin}\beta t \\ b^*=\frac{1}{N\underline{\beta}}\int_\Omega \varphi(M)F(M)d\Omega, \end{cases}$$

$$\begin{cases} W_{-B^2\Delta\varphi}(M,t)=\sum b^{**}F(M)e^{\alpha t}\underline{\sin}\beta t \\ b^{**}=\frac{1}{N\underline{\beta}}\int_\Omega -B^2\Delta\varphi(M)F(M)d\Omega, \end{cases}$$

where $F(M)$, α, β, λ, and N have been defined in Eqs. (3.33)–(3.36). Therefore, we wish to prove that

$$\lambda\int_\Omega \varphi(M)F(M)d\Omega=-\int_\Omega \Delta\varphi(M)F(M)d\Omega, \tag{3.41}$$

where

$$\lambda=\begin{cases}\lambda_n=-\dfrac{X''_n}{X_n}, & \text{one-dimensional case}\\ \lambda_n+\lambda_m=-\left(\dfrac{X''_n}{X_n}+\dfrac{Y''_m}{Y_m}\right), & \text{two-dimensional case}\\ \lambda_n+\lambda_m+\lambda_k=\left(\dfrac{X''_n}{X_n}+\dfrac{Y''_m}{Y_m}+\dfrac{Z''_k}{Z_k}\right), & \text{three-dimensional case.}\end{cases}$$

Noting that for well-posed problems, $\varphi(M)$ also satisfies the separable boundary conditions, so that it can be written as

$$\varphi(M)=\begin{cases}\varphi(x), & \text{one-dimensional case}\\ \varphi_1(x)\varphi_2(y), & \text{two-dimensional case}\\ \varphi_1(x)\varphi_2(y)\varphi_3(z), & \text{three-dimensional case.}\end{cases}$$

Thus,

$$\Delta\varphi(M)=\begin{cases}\varphi''(x), & \text{one-dimensional case}\\ \varphi''_1(x)\varphi_2(y)+\varphi''_2(y)\varphi_1(x), & \text{two-dimensional case}\\ \varphi''_1(x)\varphi_2(y)\varphi_3(z)+\varphi''_2(y)\varphi_1(x)\varphi_3(z)+\varphi''_3(z)\varphi_1(x)\varphi_2(y), & \text{three-dimensional case.}\end{cases}$$

In the following, we prove Eq. (3.41) for 1D, 2D, and 3D cases successively.

For *one-dimensional case*,

$$\begin{aligned}\lambda\int_\Omega\varphi(M)F(M)d\Omega &= -\frac{X''_n}{X_n}\int_0^{l_1}\varphi(x)X_n(x)dx\\ &= -\int_0^{l_1}X''_n(x)\varphi(x)dx\\ &= \int_0^{l_1}X'_n(x)\varphi'(x)dx - X'_n(x)\varphi(x)|_0^{l_1}\\ &= [X_n(x)\varphi'(x)-X'_n(x)\varphi(x)]|_0^{l_1}-\int_0^{l_1}X_n(x)\varphi''(x)dx\\ &= -\int_0^{l_1}X_n(x)\varphi''(x)dx = -\int_\Omega\Delta\varphi(M)F(M)d\Omega,\end{aligned}$$

where the relation

$$[X_n(x)\varphi'(x)-X'_n(x)\varphi(x)]|_{x=0,l_1}=0$$

has been used since both $X_n(x)$ and $\varphi(x)$ satisfy the same linear homogeneous boundary conditions at $x=0$ and $x=l_1$.

For *two-dimensional case*,

$$\begin{aligned}\lambda\int_\Omega\varphi(M)F(M)d\Omega &= -\left(\frac{X''_n}{X_n}+\frac{Y''_m}{Y_m}\right)\int_0^{l_2}\int_0^{l_1}\varphi_1(x)\varphi_2(y)X_n(x)Y_m(y)dxdy\\ &= -\int_0^{l_1}\varphi_1(x)X''_n(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy-\int_0^{l_2}\varphi_2(y)Y''_m(y)dy\int_0^{l_1}\varphi_1(x)X_n(x)dx\\ &= \left\{[X_n(x)\varphi'_1(x)-X'_n(x)\varphi_1(x)]|_0^{l_1}-\int_0^{l_1}X_n(x)\varphi''_1(x)dx\right\}\int_0^{l_2}\varphi_2(y)Y_m(y)dy\\ &\quad+\left\{[Y_m(y)\varphi'_2(y)-Y'_m(y)\varphi_2(y)]|_0^{l_2}-\int_0^{l_2}Y_m(y)\varphi''_2(y)dy\right\}\int_0^{l_1}\varphi_1(x)X_n(x)dx\\ &= -\int_0^{l_1}X_n(x)\varphi''_1(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy-\int_0^{l_2}Y_m(y)\varphi''_2(y)dy\int_0^{l_1}\varphi_1(x)X_n(x)dx\\ &= -\int_0^{l_2}\int_0^{l_1}[\varphi''_1(x)\varphi_2(y)+\varphi''_2(y)\varphi_1(x)]X_n(x)Y_m(y)dxdy\\ &= -\int_\Omega\Delta\varphi(M)F(M)d\Omega,\end{aligned}$$

where the relations

$$[X_n(x)\varphi'_1(x) - X'_n(x)\varphi_1(x)]|_{x=0,l_1} = 0,$$
$$[Y_m(y)\varphi'_2(y) - Y'_m(y)\varphi_2(y)]|_{y=0,l_2} = 0$$

have been used because $F(M) = X_n(x)Y_m(y)$ and $\varphi(M) = \varphi_1(x)\varphi_2(y)$ satisfy the same linear homogeneous boundary conditions in both x- and y-directions.

For *three-dimensional case*,

$$\begin{aligned}
\lambda\int_\Omega \varphi(M)F(M)d\Omega &= -\left(\frac{X''_n}{X_n}+\frac{Y''_m}{Y_m}+\frac{Z''_k}{Z_k}\right)\int_0^{l_3}\int_0^{l_2}\int_0^{l_1}\varphi_1(x)\varphi_2(y)\varphi_3(z)X_n(x)Y_m(y)Z_k(z)dxdydz\\
&= -\int_0^{l_1}\varphi_1(x)X''_n(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad -\int_0^{l_2}\varphi_2(y)Y''_m(y)dy\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad -\int_0^{l_3}\varphi_3(z)Z''_k(z)dz\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_2}\varphi_2(y)Y''_m(y)dy\\
&= \left\{[X_n(x)\varphi'_1(x) - X'_n(x)\varphi_1(x)]|_0^{l_1} - \int_0^{l_1}X_n(x)\varphi''_1(x)dx\right\}\int_0^{l_2}\varphi_2(y)Y_m(y)dy\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad +\left\{[Y_m(y)\varphi'_2(y) - Y'_m(y)\varphi_2(y)]|_0^{l_2} - \int_0^{l_2}Y_m(y)\varphi''_2(y)dy\right\}\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad +\left\{[Z_k(z)\varphi'_3(z) - Z'_k(z)\varphi_3(z)]|_0^{l_3} - \int_0^{l_3}Z_k(z)\varphi''_3(z)dz\right\}\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy\\
&= -\int_0^{l_1}X_n(x)\varphi''_1(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad -\int_0^{l_2}Y_m(y)\varphi''_2(y)dy\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_3}\varphi_3(z)Z_k(z)dz\\
&\quad -\int_0^{l_3}Z_k(z)\varphi''_3(z)dz\int_0^{l_1}\varphi_1(x)X_n(x)dx\int_0^{l_2}\varphi_2(y)Y_m(y)dy\\
&= -\int_0^{l_3}\int_0^{l_2}\int_0^{l_1}[\varphi''_1(x)\varphi_2(y)\varphi_3(z)+\varphi''_2(y)\varphi_1(x)\varphi_3(z)+\varphi''_3(z)\varphi_1(x)\varphi_2(y)]X_n(x)Y_m(y)Z_k(z)dxdydz\\
&= -\int_\Omega \Delta\varphi(M)F(M)d\Omega,
\end{aligned}$$

where the relations

$$[X_n(x)\varphi'_1(x) - X'_n(x)\varphi_1(x)]|_{x=0,l_1} = 0,$$
$$[Y_m(y)\varphi'_2(y) - Y'_m(y)\varphi_2(y)]|_{y=0,l_2} = 0,$$
$$[Z_k(k)\varphi'_3(z) - Z'_k(z)\varphi_3(z)]|_{z=0,l_3} = 0$$

have been used because $F(M) = X_n(x)Y_m(y)Z_k(z)$ and $\varphi(M) = \varphi_1(x)\varphi_2(y)\varphi_3(z)$ satisfy the same linear homogeneous boundary conditions in x-, y-, and z-directions. Therefore, we have proved Theorem 1′ for 1D, 2D, and 3D cases.

In summary, the solution of the well-posed problem

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T + B^2\dfrac{\partial}{\partial t}\Delta T + f(M,t);\ \Omega\times(0,+\infty)\\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial n}\right)\Big|_{\partial\Omega} = 0;(0,+\infty)\\ T(M,0)=\varphi(M),\quad \frac{\partial T}{\partial t}(M,0)=\psi(M);\quad \Omega\end{cases} \tag{3.42}$$

can also be written as

$$T(M,t) = W_\psi(M,t) + \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(M,t) + W_{-B^2\Delta\varphi}(M,t) + \int_0^t W_{f_\tau}(M,t-\tau)d\tau, \tag{3.43}$$

where $W_\psi(M,t)$ is the solution at $f=\varphi=0$ with the form of (3.32).

2. Polar coordinates

Boundary conditions of all three types for mixed problems in a circular domain become separable with respect to the spatial variables in a two-dimensional polar coordinate system. In this section we focus on the solution of the mixed problem:

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2 \Delta T(r,\theta,t) + B^2 \dfrac{\partial}{\partial t}\Delta T(r,\theta,t) + f(r,\theta,t); \mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; (0,+\infty) \\ T(r,\theta,0) = \varphi(r,\theta), \quad \dfrac{\partial T}{\partial t}(r,\theta,0) = \psi(r,\theta); \mathrm{D}, \end{cases} \tag{3.44}$$

where D denotes the circular domain: $0 < r < r_0$ and $0 < \theta < 2\pi$. Δ is the Laplacian in polar coordinates: $\partial^2/\partial r^2 + (1/r)\partial/\partial r + \left(1/r^2\right)\partial^2/\partial\theta^2$. $\mathrm{L}(T, \partial T/\partial r)|_{r=r_0} = 0$ encompasses all three types of linear homogeneous boundary conditions, with $\partial T/\partial r$ standing for the normal derivative on the circle $r = r_0$. The linearity of (3.44) ensures that its solution is the superposition of three solutions from $\psi(r,\theta)$, $\varphi(r,\theta)$, and $f(r,\theta,t)$, respectively. Following a similar procedure in Sec. III A 1, we first develop the solution of the $\psi-$ contribution problem in polar coordinates, and then give and prove the theorems that relate the solution of the $\psi-$ contribution problem with solutions of the $\varphi-$ and $f-$ contribution problems.

The solution from $\psi(r,\theta)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2 \Delta T(r,\theta,t) + B^2 \dfrac{\partial}{\partial t}\Delta T(r,\theta,t); \mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; \\ T(r,\theta,0) = 0, \quad \frac{\partial T}{\partial t}(r,\theta,0) = \psi(r,\theta). \end{cases} \tag{3.45}$$

Considering a solution with the form of $T(r,\theta,t) = v(r,\theta)\Gamma(t)$ and substituting it into the equation in (3.45) yield, with $-\lambda$ as the separation constant,

$$\frac{\Gamma''(t) + \frac{1}{\tau_0}\Gamma'(t)}{A^2\Gamma(t) + B^2\Gamma'(t)} = \frac{\Delta v(r,\theta)}{v(r,\theta)} = -\lambda.$$

Thus, we arrive at

$$\Gamma''(t) + \left(\frac{1}{\tau_0} + \lambda B^2\right)\Gamma'(t) + \lambda A^2 \Gamma(t) = 0 \tag{3.46}$$

and

$$\begin{cases} \Delta v(r,\theta) + \lambda v(r,\theta) = 0 \text{ or } \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} + \lambda v = 0 \\ \mathrm{L}\left(v, \dfrac{\partial v}{\partial r}\right)\Big|_{r=r_0} = 0, v(r,\theta+2\pi) = v(r,\theta), \end{cases} \tag{3.47}$$

where $v(r,\theta+2\pi) = v(r,\theta)$ is a natural boundary condition. Further, assuming a solution of (3.47) with the form of $v(r,\theta) = R(r)\Theta(\theta)$, (3.47) becomes, with $-\mu$ as the separation constant,

$$\frac{R'' + \frac{1}{r}R' + \lambda R}{-\frac{R}{r^2}} = \frac{\Theta''}{\Theta} = -\mu.$$

Therefore,

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{\mu}{r^2}\right)R = 0, \mathrm{L}(R, R')|_{r=r_0} = 0, \tag{3.48}$$

$$\Theta'' + \mu\Theta = 0, \Theta(\theta + 2\pi) = \Theta(\theta). \tag{3.49}$$

The problem (3.49) has the solution of

$$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta \tag{3.50}$$

with $\mu = n^2, n = 0, 1, 2, ...$ to satisfy the periodic condition $\Theta(\theta + 2\pi) = \Theta(\theta)$.

Substituting $\mu = n^2$ into Eq. (3.48) yields an *eigenvalue problem of Bessel equations* with $\lambda_n > 0$:

$$\begin{cases} R''_n + \dfrac{1}{r}R'_n + \left(\lambda_n - \dfrac{n^2}{r^2}\right)R_n = 0 \\ \mathrm{L}(R_n, R'_n)|_{r=r_0} = 0, |R_n(0)| < \infty, |R'_n(0)| < \infty, \end{cases} \tag{3.51}$$

where the bounded conditions $|R_n(0)| < \infty$ and $|R'_n(0)| < \infty$ are another kind of natural boundary condition. The general solution of (3.51) is

$$R_n(r) = c_n J_n(\sqrt{\lambda_n} r) + d_n Y_n(\sqrt{\lambda_n} r),$$

where c_n and d_n are arbitrary constants. J_n and Y_n are the n-th order Bessel functions of the first and the second kinds, respectively. Applying the bounded condition $|R_n(0)| < \infty$ and $\lim_{r\to 0} Y_n(\sqrt{\lambda_n} r) = \infty$, we obtain $d_n = 0$. Therefore, the eigenfunctions and eigenvalues of (3.51) for three types of linear homogeneous boundary conditions are

$$J_n(\sqrt{\lambda_{nm}} r), \quad \lambda_{nm} = \left(\mu_m^{(n)}/r_0\right)^2, \tag{3.52}$$

where $\mu_m^{(n)}$ is the m-th ($m = 1, 2, 3, ...$) positive zero point of

$$\begin{cases} J_n(x); & \text{1st type of boundary condition: } R(r_0) = 0; \\ J'_n(x); & \text{2nd type of boundary condition: } R'(r_0) = 0; \mu_1^{(0)} = 0; \\ \dfrac{1}{r_0} x J'_n(x) + h J_n(\mathrm{x}); & \text{3rd type of boundary condition: } R'(r_0) + hR(r_0) = 0. \end{cases} \tag{3.53}$$

With these eigenvalues λ_{nm}, we obtain the solution of the (3.47)

$$v_{nm}(r,\theta) = (a_{nm}\cos n\theta + b_{nm}\sin n\theta)J_n(\sqrt{\lambda_{nm}}r),$$

where a_{nm} and b_{nm} are not all zero. We can also have the characteristic roots of Eq. (3.46):

$$r_{1,2} = \frac{1}{2}\left[-\left(\frac{1}{\tau_0}+\lambda_{nm}B^2\right)\pm\sqrt{\left(\frac{1}{\tau_0}+\lambda_{nm}B^2\right)^2 - 4\lambda_{nm}A^2}\right] = \alpha_{nm}\pm\beta_{nm}\mathbf{i},$$

$$\alpha_{nm} = -\frac{1}{2}\left(\frac{1}{\tau_0}+\lambda_{nm}B^2\right), \tag{3.54}$$

$$\beta_{nm} = \frac{1}{2}\sqrt{4\lambda_{nm}A^2 - \left(\frac{1}{\tau_0}+\lambda_{nm}B^2\right)^2}. \tag{3.55}$$

Therefore, the $T(r,\theta,t)$ that satisfies the equation and boundary conditions of (3.45) reads

$$T(r,\theta,t) = \sum_{n=0,m=1}^{+\infty} e^{\alpha_{nm}t}\left[\left(a_{nm}^*\cos\beta_{nm}t + b_{nm}^*\underline{\sin}\beta_{nm}t\right)\cos n\theta + \left(c_{nm}^*\cos\beta_{nm}t + d_{nm}^*\underline{\sin}\beta_{nm}t\right)\sin n\theta\right]J_n(\sqrt{\lambda_{nm}}r).$$

The initial condition $T(r,\theta,0)=0$ leads to $a_{nm}^* = c_{nm}^* = 0$. b_{nm}^* and d_{nm}^* can be determined to satisfy the initial condition $\partial T(r,\theta,0)/\partial t = \psi(r,\theta)$. Finally, we obtain the solution of the $\psi-$ contribution problem (3.45):

$$\begin{cases} W_\psi(r,\theta,t) = \sum_{n=0,m=1}^{\infty}\left(b_{nm}^*\cos n\theta + d_{nm}^*\sin n\theta\right)J_n(\sqrt{\lambda_{nm}}r)e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t \\ b_{nm}^* = \dfrac{1}{N_0N_{nm}\underline{\beta}_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\psi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\cos n\theta drd\theta \\ d_{nm}^* = \dfrac{1}{N_0N_{nm}\underline{\beta}_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\psi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\sin n\theta drd\theta, \end{cases} \tag{3.56}$$

where N_0 is the normal square of $\{1,\cos\theta,\sin\theta,...,\cos n\theta,\sin n\theta,...\}$:

$$N_0 = \begin{cases} 2\pi, & n=0 \\ \pi, & n>0. \end{cases} \tag{3.57}$$

$N_{nm} = \int_0^{r_0} J_n^2(\sqrt{\lambda_{nm}}r)rdr$ is the normal square of $\left\{J_n(\sqrt{\lambda_{nm}}r)\right\}$ with the value of:

$$\begin{cases} \dfrac{r_0^2}{2}J_{n+1}^2(\mu_m^{(n)}),\ \mu_m^{(n)} \text{ are zero points of } J_n(x), m=1,2,3,... \\ \quad\text{for the 1st type of boundary condition: } R(r_0)=0; \\ \dfrac{r_0^2}{2}\left[1-\left(\dfrac{n}{\mu_m^{(n)}}\right)^2\right]J_n^2(\mu_m^{(n)}),\ \mu_m^{(n)} \text{ are zero points of } J'_n(x), m=1,2,3,... \\ \quad\text{for the 2nd type of boundary condition: } R'(r_0)=0;\ \mu_1^{(0)}=0; \\ \dfrac{r_0^2}{2}\left[1+\dfrac{(r_0h)^2-n^2}{\left(\mu_m^{(n)}\right)^2}\right]J_n^2\left(\mu_m^{(n)}\right),\ \mu_m^{(n)} \text{ are zero points of } \dfrac{1}{r_0}xJ'_n(x)+hJ_n(\mathrm{x}), m=1,2,3,... \\ \quad\text{for the 3rd type of boundary condition: } R'(r_0)+hR(r_0)=0. \end{cases} \tag{3.58}$$

When T is only a function of r and t (not θ), the solution structure (3.56) still holds with n being the constant 0.

Now we develop Theorems 3 and 4 that relate the solution of $\psi-$ contribution problem with solutions of the $\varphi-$ and $f-$ contribution problems.

Theorem 3. Let $W_\psi(r,\theta,t)$ be the solution of $\psi-$ contribution problem (3.45). The solution of $\varphi-$ contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,t)+B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,t);\mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; \\ T(r,\theta,0)=\varphi(r,\theta),\quad \dfrac{\partial T}{\partial t}(r,\theta,0)=0 \end{cases} \tag{3.59}$$

is

$$T(r,\theta,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(r,\theta)+B^2W_{\lambda_{nm}\varphi}(r,\theta), \qquad (3.60)$$

where $\lambda_{nm}=\left(\mu_m^{(n)}/r_0\right)^2$ with $\mu_m^{(n)}$ being the positive zero points of (3.53).

Proof. By following a similar approach in solving (3.45), we obtain $T(r,\theta,t)$ satisfying the equation and boundary conditions of (3.59) with a form of

$$T(r,\theta,t)=\sum_{n=0,m=1}^{+\infty}e^{\alpha_{nm}t}[(a_{nm}\cos\beta_{nm}t+b_{nm}\underline{\sin}\beta_{nm}t)\cos n\theta +(c_{nm}\cos\beta_{nm}t+d_{nm}\underline{\sin}\beta_{nm}t)\sin n\theta]J_n(\sqrt{\lambda_{nm}}r). \qquad (3.61)$$

Applying the initial condition $T(r,\theta,0)=\varphi(r,\theta)$ yields

$$\begin{cases} a_{nm}=\dfrac{1}{N_0N_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\cos n\theta drd\theta \\ c_{nm}=\dfrac{1}{N_0N_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\sin n\theta drd\theta. \end{cases}$$

Applying the initial condition $\partial T/\partial t(r,\theta,0)=0$ leads to

$$\begin{cases} a_{nm}\alpha_{nm}+b_{nm}\underline{\beta}_{nm}=0,\ \text{or}\ b_{nm}=-\dfrac{\alpha_{nm}}{\underline{\beta}_{nm}}a_{nm} \\ c_{nm}\alpha_{nm}+d_{nm}\underline{\beta}_{nm}=0,\ \text{or}\ d_{nm}=-\dfrac{\alpha_{nm}}{\underline{\beta}_{nm}}c_{nm}. \end{cases}$$

Here λ_{nm}, α_{nm}, β_{nm}, N_0, and N_{nm} are given by Eqs. (3.52), (3.54), (3.55), (3.57), and (3.58), respectively. Comparing a_{nm}, b_{nm}, c_{nm}, and d_{nm} with b^*_{nm} and d^*_{nm} in (3.56), we have

$$a_{nm}=\underline{\beta}_{nm}b^{**}_{nm},\quad b_{nm}=-\alpha_{nm}b^{**}_{nm},$$

$$c_{nm}=\underline{\beta}_{nm}d^{**}_{nm},\quad d_{nm}=-\alpha_{nm}d^{**}_{nm},$$

where b^{**}_{nm} and d^{**}_{nm} have the same structures with b^*_{nm} and d^*_{nm} with $\psi(r,\theta)$ replaced with $\varphi(r,\theta)$,

$$\begin{cases} b^{**}_{nm}=\dfrac{1}{N_0N_{nm}\underline{\beta}_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\cos n\theta drd\theta \\ d^{**}_{nm}=\dfrac{1}{N_0N_{nm}\underline{\beta}_{nm}}\displaystyle\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta)J_n(\sqrt{\lambda_{nm}}r)r\sin n\theta drd\theta. \end{cases} \qquad (3.62)$$

Substituting these relations into (3.61) yields

$$T(r,\theta,t)=\sum_{n=0,m=1}^{+\infty}e^{\alpha_{nm}t}\left[\left(\underline{\beta}_{nm}b^{**}_{nm}\cos\beta_{nm}t-\alpha_{nm}b^{**}_{nm}\underline{\sin}\beta_{nm}t\right)\times\cos n\theta+\left(\underline{\beta}_{nm}d^{**}_{nm}\cos\beta_{nm}t-\alpha_{nm}d^{**}_{nm}\underline{\sin}\beta_{nm}t\right)\times\sin n\theta\right]J_n(\sqrt{\lambda_{nm}}r). \qquad (3.63)$$

By the structure of $W_{\psi}(r,\theta,t)$ in (3.56), we have

$$\frac{\partial W_{\varphi}(r,\theta,t)}{\partial t}=\sum_{n=0,m=1}^{\infty}\left(b^{**}_{nm}\cos n\theta+d^{**}_{nm}\sin n\theta\right)\times\left(\alpha_{nm}\underline{\sin}\beta_{nm}t+\underline{\beta}_{nm}\cos\beta_{nm}t\right)e^{\alpha_{nm}t}J_n(\sqrt{\lambda_{nm}}r).$$

Also,

$$\frac{1}{\tau_0}W_{\varphi}(r,\theta,t)+B^2W_{\lambda_{nm}\varphi}(r,\theta,t)=\sum_{n=0,m=1}^{\infty}\left(\frac{1}{\tau_0}+\lambda_{nm}B^2\right)\times\left(b^{**}_{nm}\cos n\theta+d^{**}_{nm}\sin n\theta\right)\times J_n(\sqrt{\lambda_{nm}}r)e^{\alpha_{nm}t}\underline{\sin}\beta_{nm}t$$

By adding $\partial W_{\varphi}/\partial t$ and $(1/\tau_0)W_{\varphi}+B^2W_{\lambda_{nm}\varphi}$, we can readily obtain that the solutions given by Eqs. (3.60) and (3.63) are exactly the same. Theorem 3 is therefore proved. Theorem 3 still holds with n being the constant 0.

Theorem 4. Let $W_{\psi}(r,\theta,t)$ be the solution of $\psi-$ contribution problem (3.45). The solution of $f-$ contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2T}{\partial t^2}=A^2\Delta T(r,\theta,t)+B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,t)+f(r,\theta,t);\mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0}=0; \\ T(r,\theta,0)=0,\quad \dfrac{\partial T}{\partial t}(r,\theta,0)=0 \end{cases} \qquad (3.64)$$

is

$$T(r,\theta,t)=\int_0^{r_0}W_{f_\tau}(r,\theta,t-\tau)d\tau, \qquad (3.65)$$

where $f_\tau=f(r,\theta,\tau)$.

Theorem 4 can be proved by following the same way as the proof of Theorem 2.

Therefore, by the principle of superposition, the solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T(r,\theta,t)+B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,t)+f(r,\theta,t);\mathrm{D}\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0}=0;(0,+\infty) \\ T(r,\theta,0)=\varphi(r,\theta),\quad \dfrac{\partial T}{\partial t}(r,\theta,0)=\psi(r,\theta);\mathrm{D} \end{cases} \tag{3.66}$$

is

$$T(r,\theta,t)=W_\psi(r,\theta,t)+\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_\varphi(r,\theta,t)+B^2W_{\lambda_{nm}\varphi}(r,\theta,t)+\int_0^{r_0}W_{f_\tau}(r,\theta,t-\tau)d\tau, \tag{3.67}$$

where $W_\psi(r,\theta,t)$ is available in (3.56). Note that Eqs. (3.56) and (3.67) are valid for all three types of boundary conditions, but $\mu_m^{(n)}$ are boundary-condition-dependent as given in (3.53).

When we use the structure of W_ψ in Eq. (3.56) and the solution structure Theorems 3 and 4 to obtain the solution of (3.44), if β_{nm} is purely imaginary for some n, m, and k, we can change $\sin\beta_{nm}t$ into $\left(\mathrm{e}^{\mathbf{i}\beta_{nm}t}-\mathrm{e}^{-\mathbf{i}\beta_{nm}t}\right)/(2\mathbf{i})$.

3. Cylindrical coordinates

In this section, we seek solutions of mixed problems in a cylindrical coordinate system,

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T(r,\theta,z,t)+B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,z,t)+f(r,\theta,z,t);\quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r},\dfrac{\partial T}{\partial z}\right)\Big|_{\partial\Omega}=0;(0,+\infty) \\ T(r,\theta,z,0)=\varphi(r,\theta,z),\quad \dfrac{\partial T}{\partial t}(r,\theta,z,0)=\psi(r,\theta,z);\quad \Omega, \end{cases} \tag{3.68}$$

where Ω stands for the cylindrical domain:$0<r<r_0$, $0\le\theta<2\pi$, $0<z<l_0$. $\partial\Omega$ is the boundary of Ω. Δ is the Laplacian in cylindrical coordinates: $\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\theta^2+\partial^2/\partial z^2$. $\mathrm{L}(T,\partial T/\partial r,\partial T/\partial z)|_{\partial\Omega}=0$ stands for a total of 27 combinations of boundary conditions of all three types. Again, we first find the solution from $\psi(r,\theta,z)$, and then examine the relation between the solution from $\psi(r,\theta,z)$ and those from $\varphi(r,\theta,z)$ and $f(r,\theta,z,t)$.

The solution due to $\psi(r,\theta,z)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T(r,\theta,z,t)+B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,z,t);\quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r},\dfrac{\partial T}{\partial z}\right)\Big|_{\partial\Omega}=0;(0,+\infty) \\ T(r,\theta,z,0)=0,\quad \dfrac{\partial T}{\partial t}(r,\theta,z,0)=\psi(r,\theta,z);\quad \Omega. \end{cases} \tag{3.69}$$

Assume $T(r,\theta,z,t)=\Gamma(t)V(r,\theta,z)$. Equation (3.69) can be written as, with $-\lambda_1$ as the separation constant,

$$\frac{\Gamma''(t)+\frac{1}{\tau_0}\Gamma'(t)}{A^2\Gamma(t)+B^2\Gamma'(t)}=\frac{\Delta V(r,\theta,z)}{V(r,\theta,z)}=-\lambda_1.$$

We thus arrive at

$$\Gamma''(t)+\left(\frac{1}{\tau_0}+\lambda_1B^2\right)\Gamma'(t)+\lambda_1A^2\Gamma(t)=0 \tag{3.70}$$

and

$$\begin{cases} \Delta V(r,\theta,z)+\lambda_1V(r,\theta,z)=0 \text{ or } \dfrac{\partial^2 V}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial V}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\theta^2}+\dfrac{\partial^2 V}{\partial z^2}+\lambda_1V=0 \\ \mathrm{L}\left(V,\dfrac{\partial V}{\partial r}\right)\Big|_{r=r_0}=0, V(r,\theta+2\pi,z)=V(r,\theta,z). \end{cases} \tag{3.71}$$

Let $V(r,\theta,z) = v(r,\theta)Z(z)$. Substituting this into (3.71) leads to

$$\frac{\Delta v(r,\theta)}{v(r,\theta)} = -\frac{Z''(z)}{Z(z)} - \lambda_1 = -\lambda_2,$$

where $-\lambda_2$ is the separation constant. We have

$$\begin{cases} \Delta v(r,\theta) + \lambda_2 v(r,\theta) = 0 \text{ or } \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} + \lambda_2 v = 0 \\ \mathrm{L}\left(v, \dfrac{\partial v}{\partial r}\right)\Big|_{r=r_0} = 0, v(r,\theta+2\pi) = v(r,\theta) \end{cases} \tag{3.72}$$

and

$$\begin{cases} Z''(z) + (\lambda_1 - \lambda_2)Z(z) = 0 \\ \mathrm{L}(Z,Z')|_{z=0} = 0, \mathrm{L}(Z,Z')|_{z=l_0} = 0. \end{cases} \tag{3.73}$$

We have obtained the solution of the eigenvalue problem (3.72) in Sec. III A 2:

$$v_{nm}(r,\theta) = (a_{nm}\cos n\theta + b_{nm}\sin n\theta)J_n(\sqrt{\lambda_{nm}}r).$$

in which a_{nm} and b_{nm} are constants. $\lambda_{nm} = \lambda_2$ are the eigenvalues, $\left(\mu_m^{(n)}/r_0\right)^2$, where $\mu_m^{(n)}$ are the zero points of (3.53).

There are a total of nine combinations of boundary conditions in the problem (3.73). Let $\lambda_k = \lambda_1 - \lambda_2$ and $Z_k(z)$ be the eigenvalues and eigenfunctions, which are available in Table II. Substitute $\lambda_1 = \lambda_{nm} + \lambda_k$ into Eq. (3.70) to obtain

$$\Gamma''(t) + \left[\frac{1}{\tau_0} + (\lambda_{nm} + \lambda_k)B^2\right]\Gamma'(t) + (\lambda_{nm} + \lambda_k)A^2\Gamma(t) = 0. \tag{3.74}$$

Its solution is

$$\Gamma_{nmk}(t) = \mathrm{e}^{\alpha_{nmk}t}(c_{nmk}\cos\beta_{nmk}t + d_{nmk}\underline{\sin}\beta_{nmk}t),$$

where c_{nmk} and d_{nmk} are constants. α_{nmk} and β_{nmk} are given by

$$\alpha_{nmk} = -\frac{1}{2}\left[\frac{1}{\tau_0} + (\lambda_{nm} + \lambda_k)B^2\right],$$

$$\beta_{nm} = \frac{1}{2}\sqrt{4(\lambda_{nm} + \lambda_k)A^2 - \left[\frac{1}{\tau_0} + (\lambda_{nm} + \lambda_k)B^2\right]^2}.$$

Therefore, the $T(r,\theta,z,t)$ that satisfies the equation and boundary conditions in (3.69) can be written as

$$\begin{aligned} T(r,\theta,z,t) = \sum_{n,m,k} e^{\alpha_{nmk}t}\big[&\left(a^*_{nmk}\cos\beta_{nmk}t + b^*_{nmk}\underline{\sin}\beta_{nmk}t\right)\cos n\theta \\ &+\left(c^*_{nmk}\cos\beta_{nmk}t + d^*_{nmk}\underline{\sin}\beta_{nmk}t\right)\sin n\theta\big] \\ &\times J_n(\sqrt{\lambda_{nm}}r)Z_k(z), \end{aligned}$$

where $\sum_{n,m,k}$ stands for a triple summation with n and m starting from 0 and 1, respectively, and k starting from either 0 or 1 based on the boundary conditions in the z-direction.

The initial condition $T(r,\theta,z,0) = 0$ yields $a^*_{nmk} = c^*_{nmk} = 0$. b^*_{nmk} and d^*_{nmk} can be determined by applying the initial condition $\partial T(r,\theta,z,0)/\partial t = \psi(r,\theta,z)$. Finally, we obtain the solution of (3.69),

$$\begin{cases} W_\psi(r,\theta,z,t) = \sum_{n,m,k}\left(b^*_{nmk}\cos n\theta + d^*_{nmk}\sin n\theta\right)J_n(\sqrt{\lambda_{nm}}r)Z_k(z)e^{\alpha_{nmk}t}\underline{\sin}\beta_{nmk}t \\ b^*_{nmk} = \dfrac{1}{N_0 N_{nm} N_k \underline{\beta}_{nmk}}\displaystyle\iiint_\Omega \psi(r,\theta,z)rJ_n(\sqrt{\lambda_{nm}}r)Z_k(z)\cos n\theta drd\theta dz \\ d^*_{nmk} = \dfrac{1}{N_0 N_{nm} N_k \underline{\beta}_{nmk}}\displaystyle\iiint_\Omega \psi(r,\theta,z)rJ_n(\sqrt{\lambda_{nm}}r)Z_k(z)\sin n\theta drd\theta dz, \end{cases} \tag{3.75}$$

where N_0, N_{nm}, and N_k are normal squares of $\{1, \cos\theta, \sin\theta, ..., \cos n\theta, \sin n\theta, ...\}$, $\left\{J_n(\sqrt{\lambda_{nm}}r)\right\}$ and $\{Z_k(z)\}$, and given by (3.57), (3.58), and Table II, respectively.

Theorem 5. Let $W_\psi(r,\theta,z,t)$ be the solution of $\psi-$contribution problem (3.69). The solution of $\varphi-$ contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,z,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T, \dfrac{\partial T}{\partial r}, \dfrac{\partial T}{\partial z}\right)\Big|_{\partial\Omega} = 0; (0,+\infty) \\ T(r,\theta,z,0) = \varphi(r,\theta,z), \quad \dfrac{\partial T}{\partial t}(r,\theta,z,0) = 0; \quad \Omega \end{cases} \tag{3.76}$$

is

$$T(r,\theta,z,t) = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(r,\theta,z) + B^2 W_{(\lambda_{nm}+\lambda_k)\varphi}(r,\theta,z), \tag{3.77}$$

where λ_{nm} and λ_k are available in Eq. (3.52) and Table II, respectively.

Theorem 5 can be readily proved by following a similar way in the proof of Theorem 3 and noting that $\alpha_{nmk} = -[1/\tau_0 + (\lambda_{nm}+\lambda_k)B^2]/2$.

Theorem 6. Let $W_{\psi}(r,\theta,z,t)$ be the solution of $\psi-$ contribution problem (3.69). The solution of $f-$ contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,z,t) + B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,z,t) + f(r,\theta,z,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r},\dfrac{\partial T}{\partial z}\right)\Big|_{\partial\Omega} = 0; (0,+\infty) \\ T(r,\theta,z,0)=0, \quad \dfrac{\partial T}{\partial t}(r,\theta,z,0)=0; \quad \Omega \end{cases} \tag{3.78}$$

is

$$T(r,\theta,z,t) = \int_0^{r_0} W_{f_\tau}(r,\theta,z,t-\tau)d\tau, \tag{3.79}$$

where $f_\tau = f(r,\theta,z,\tau)$.

Theorem 6 can be proved by following the same way as the proof of Theorem 2.

By the structure of W_{ψ} in Eq. (3.75), the $T(r,\theta,z,t)$ in Eq. (3.79) reads

$$\begin{aligned} T(r,\theta,z,t) &= \int_0^t W_{f_\tau}(r,\theta,z,t-\tau)d\tau \\ &= \int_0^t \iiint_{\Omega} \sum_{n,m,k}\frac{1}{N_0 N_{nm} N_k \underline{\beta}_{nmk}} e^{\alpha_{nmk}(t-\tau)} \\ &\quad \times f(r^*,\theta^*,z^*,\tau)(\cos n\theta\cos n\theta^* + \sin n\theta \sin n\theta^*) \\ &\quad \times J_n(\sqrt{\lambda_{nm}}r) \\ &\quad \times r^* J_n(\sqrt{\lambda_{nm}}r^*)Z_k(z)Z_k(z^*) \\ &\quad \times \underline{\sin}\beta_{nmk}(t-\tau)dr^* d\theta^* dz^* d\tau \\ &= \int_0^t \iiint_{\Omega} G(r,r^*;\theta,\theta^*;z,z^*;t-\tau) \\ &\quad \times f(r^*,\theta^*,z^*,\tau)d\Omega d\tau, \end{aligned} \tag{3.80}$$

where

$$\begin{aligned} G(r,r^*;\theta,\theta^*;z,z^*;t-\tau) &= \sum_{n,m,k}\frac{1}{N_0 N_{nm} N_k \underline{\beta}_{nmk}} e^{\alpha_{nmk}(t-\tau)} \\ &\quad \times J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nm}}r^*)r^* \\ &\quad \times Z_k(z)Z_k(z^*)\cos n(\theta-\theta^*) \\ &\quad \times \underline{\sin}\beta_{nmk}(t-\tau) \end{aligned} \tag{3.81}$$

is called the *Green function of the dual-phase-lagging heat conduction equation in a cylindrical domain*. The Green function is clearly boundary-condition dependent. When $f(r,\theta,z,t) = \delta(\mathbf{r}-\mathbf{r}_0, t-t_0)$, the solution of (3.78) reduces to

$$T(r,\theta,z,t) = G(r,r_0;\theta,\theta_0;z,z_0;t-t_0),$$

where $\mathbf{r} = (r,\theta,z)$, $\mathbf{r}_0 = (r_0,\theta_0,z_0)$. Therefore the Green function $G(r,r_0;\theta,\theta_0;z,z_0;t-t_0)$ is the solution from the source term $\delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$.

According to Theorems 5 and 6, the solution of (3.68) is

$$\begin{aligned} T(r,\theta,z,t) &= W_{\psi}(r,\theta,z) + \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(r,\theta,z) \\ &\quad + B^2 W_{(\lambda_{nm}+\lambda_k)\varphi}(r,\theta,z) + \int_0^{r_0} W_{f_\tau}(r,\theta,z,t-\tau)d\tau, \end{aligned} \tag{3.82}$$

where $W_{\psi}(r,\theta,z)$ is given by (3.75). Therefore, we can directly write out the solution of (3.68) based on (3.75) and (3.82) without going through all the details. If β_{nmk} is purely imaginary for some n, m, and k, we can change $\sin\beta_{nmk}t$ into $(\mathrm{e}^{\mathbf{i}\beta_{nmk}t} - \mathrm{e}^{-\mathbf{i}\beta_{nmk}t})/(2\mathbf{i})$.

4. Spherical coordinates

Boundary conditions of all three types for mixed problems in a spherical domain become separable with respect to the spatial variables in spherical coordinate system. In this section, we develop the solution of the mixed problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,\phi,t) + B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,\phi,t) + f(r,\theta,\phi,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; (0,+\infty) \\ T(r,\theta,\phi,0)=\varphi(r,\theta,\phi), \quad \dfrac{\partial T}{\partial t}(r,\theta,\phi,0)=\psi(r,\theta,\phi); \quad \Omega, \end{cases} \tag{3.83}$$

where Ω stands for a sphere of radius r_0, with $\partial\Omega$ as its boundary. θ is the azimuthal angle:$0 < \theta < 2\pi$, ϕ is the polar angle:$0 < \phi < \pi$. Δ is the Laplacian in spherical coordinates:

$$\begin{aligned}\Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{\partial}{\partial\phi} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.\end{aligned}$$

The boundary condition $\mathrm{L}(T, \partial T/\partial r)|_{r=r_0} = 0$ includes all three types. Again, we first seek the solution $W_\psi(r,\theta,\phi,t)$ that satisfies the problem:

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,\phi,t) + B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,\phi,t); \quad \Omega\times(0,+\infty)\\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0;\\ T(r,\theta,\phi,0) = 0, \quad \dfrac{\partial T}{\partial t}(r,\theta,\phi,0) = \psi(r,\theta,\phi);\end{cases} \tag{3.84}$$

Assume $T = \Gamma(t)V(r,\theta,\phi)$ and substitute it into the equation of (3.84). We obtain the two eigenvalue problems with λ as the separation constant

$$\Gamma''(t) + \left(\frac{1}{\tau_0} + \lambda B^2\right)\Gamma'(t) + \lambda A^2\Gamma(t) = 0 \tag{3.85}$$

and

$$\begin{cases}\Delta V(r,\theta,\phi) + \lambda V(r,\theta,\phi) = 0\\ \text{or } \dfrac{\partial^2 V}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2\sin^2\phi}\dfrac{\partial^2 V}{\partial\theta^2} + \dfrac{\cos\phi}{r^2\sin\phi}\dfrac{\partial V}{\partial\phi} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\phi^2} + \lambda V(r,\theta,\phi) = 0\\ \mathrm{L}\left(v,\dfrac{\partial V}{\partial r}\right)\Big|_{r=r_0} = 0, V(r,\theta+2\pi,\phi) = v(r,\theta,\phi).\end{cases} \tag{3.86}$$

Further separate $V(r,\theta,\phi)$ into $R(r)Y(\theta,\phi)$ and substitute it into the equation of (3.86). We can obtain that

$$r^2R'' + 2rR' + \left[\lambda r^2 - l(l+1)\right]R = 0, \quad \mathrm{L}(R,R')|_{r=r_0} = 0 \tag{3.87}$$

and

$$\begin{gathered}\frac{1}{\sin^2\phi}\frac{\partial^2 Y}{\partial\theta^2} + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y}{\partial\phi}\right) + l(l+1)\\ Y = 0, \quad Y(\theta+2\pi,\phi) = Y(\theta,\phi)\end{gathered} \tag{3.88}$$

in which the separation constant is customarily denoted as $l(l+1)$. Assume $Y = \Theta(\theta)\Phi(\phi)$. Substituting it into Eq. (3.88) leads to

$$\Theta'' + \eta\Theta = 0, \quad \Theta(\theta+2\pi) = \Theta(\theta) \tag{3.89}$$

and

$$\begin{gathered}\Phi'' + (\cot\phi)\Phi' + \left[l(l+1) - \frac{\eta}{\sin^2\phi}\right]\Phi = 0,\\ 0 < \phi < \pi, \quad |\Phi(\phi)| < \infty,\end{gathered} \tag{3.90}$$

where η is the separation constant.

The eigenvalue problem (3.89) has the solution of

$$\Theta_n(\theta) = a_n\cos n\theta + b_n\sin n\theta \tag{3.91}$$

with $\eta = n^2, n = 0, 1, 2,...$ to satisfy the periodic condition $\Theta(\theta+2\pi) = \Theta(\theta)$. a_n and b_n are arbitrary constants and cannot be all zero. Also, $a_0 \neq 0$.

With $\eta = n^2$, Eq. (3.90) forms an *eigenvalue problem of the Legendre equation* when the eigenvalues $l(l+1) = m(m+1)$. Its solution is

$$\begin{cases} \Phi_m(\phi) = P_m(\cos\phi), l(l+1) = m(m+1), m = 0, 1, 2, \ldots; & \text{when } n = 0 \\ \Phi_m(\phi) = P_m^n(\cos\phi), l(l+1) = m(m+1), m = 1, 2, 3, \ldots \text{ and } n \le m; & \text{when } n > 0, \end{cases} \tag{3.92}$$

where $P_m(\cos\theta)$ is the *Legendre polynomial of degree m*; $P_m^n(\cos\theta)$ is the *Associated Legendre polynomial of degree m and order n*.

With $l = m = 0, 1, 2, \ldots$, Eq. (3.87) together with $|R(0)| < \infty$ and $|R'(0)| < \infty$ forms another eigenvalue problem. To solve this problem, define a new variable $x = \lambda^{1/2} r$ and a new function $y(x) = x^{1/2} R$. Equation (3.87) thus transforms into a Bessel equation,

$$x^2 y'' + x y' + \left[x^2 - \left(m + \frac{1}{2}\right)^2\right] y = 0,$$

whose solution is

$$R_m(r) = \frac{1}{\sqrt{\lambda^{1/2} r}} \left[c_m J_{m+\frac{1}{2}}\left(\lambda^{1/2} r\right) + d_m J_{-(m+\frac{1}{2})}\left(\lambda^{1/2} r\right)\right],$$

where c_m and d_m are constants that are not all zero. To satisfy $|R(0)| < \infty$, we have $d_m = 0$ and $c_m \neq 0$. Without taking account of a constant factor, $R_m(r)$ can be written as

$$R_m(r) = \sqrt{\frac{\pi}{2\lambda^{1/2} r}} J_{m+\frac{1}{2}}\left(\lambda^{1/2} r\right) = j_m\left(\lambda^{1/2} r\right).$$

Here $j_m(x) = \sqrt{\pi/(2x)} J_{m+1/2}(x)$ is the *spherical Bessel function of the first kind*. The eigenvalues are thus

$$\lambda_{mk} = \left(\mu_k^{(m+\frac{1}{2})} / r_0\right)^2, \quad m = 0, 1, 2, \ldots, \; k = 1, 2, 3, \ldots$$

Eigenfunctions:

$$j_m(\lambda_{mk} r) = j_m\left(\mu_k^{(m+\frac{1}{2})} / r_0\right),$$

where $\mu_k^{(m+1/2)}$ are the k-th positive zero point of

$$\begin{cases} J_{m+\frac{1}{2}}(x); & \text{1st type of boundary condition: } R(r_0) = 0; \\ x J'_{m+\frac{1}{2}}(x) - \frac{1}{2} J_{m+\frac{1}{2}}(x); & \text{2nd type of boundary condition: } R'(r_0) = 0; \\ x J'_{m+\frac{1}{2}}(x) + \left(h r_0 - \frac{1}{2}\right) J_{m+\frac{1}{2}}(x); & \text{3rd type of boundary condition:} \\ & R'(r_0) + h R(r_0) = 0. \end{cases} \tag{3.93}$$

Therefore, we obtain the solution of the $V(r, \theta, \phi)$ − problem (3.86):

$$V_{nmk}(r, \theta, \phi) = (a_{nmk} \cos n\theta + b_{nmk} \sin n\theta) P_m^n(\cos\theta) j_m(\lambda_{mk}^{1/2} r), \quad n \le m.$$

The characteristic roots of the $\Gamma(t)$-function (3.85) are

$$r_{1,2} = \frac{1}{2}\left[-\left(\frac{1}{\tau_0} + \lambda_{mk} B^2\right) \pm \sqrt{\left(\frac{1}{\tau_0} + \lambda_{mk} B^2\right)^2 - 4\lambda_{mk} A^2}\right] = \alpha_{mk} \pm \beta_{mk} \mathbf{i},$$

$$\alpha_{mk} = -\frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_{mk} B^2\right), \tag{3.94}$$

$$\beta_{mk} = \frac{1}{2}\sqrt{4\lambda_{mk} A^2 - \left(\frac{1}{\tau_0} + \lambda_{mk} B^2\right)^2}, \tag{3.95}$$

so that

$$\Gamma_{mk}(t) = e^{\alpha_{mk} t} (c_{mk} \cos \beta_{mk} t + d_{mk} \underline{\sin} \beta_{mk} t).$$

Thus, the $T(r, \theta, \phi, t)$ that satisfies the equation and boundary conditions of (3.84) is

$$T(r, \theta, \phi, t) = \sum_{n,m,k} e^{\alpha_{mk} t} \left[\left(a^*_{nmk} \cos \beta_{mk} t + b^*_{nmk} \underline{\sin} \beta_{mk} t\right) \cos n\theta + \left(c^*_{nmk} \cos \beta_{mk} t + d^*_{nmk} \underline{\sin} \beta_{mk} t\right) \sin n\theta\right] \times P_m^n(\cos\phi) j_m(\lambda_{mk}^{1/2} r). \tag{3.96}$$

The initial condition $T(r, \theta, \phi, t) = 0$ yields $a^*_{nmk} = c^*_{nmk} = 0$. b^*_{nmk} and d^*_{nmk} are determined by the initial condition $\partial T / \partial t|_{t=0} = 0$. Finally, we obtain the solution structure of W_ψ satisfying the problem (3.84),

$$\begin{cases} W_\psi(r, \theta, \phi, t) = \sum_{n=0, m=0, k=1}^{\infty} \left(b^*_{nmk} \cos n\theta + d^*_{nmk} \sin n\theta\right) P_m^n(\cos\phi) j_m(\sqrt{\lambda_{mk}} r) e^{\alpha_{mk} t} \underline{\sin} \beta_{mk} t \\ b^*_{nmk} = \dfrac{1}{N_0 N_{nm} N_{mk} \underline{\beta}_{mk}} \displaystyle\int_0^\pi \int_{-\pi}^\pi \int_0^{r_0} \psi(r, \theta, \phi) P_m^n(\cos\phi) j_m(\sqrt{\lambda_{mk}} r) r^2 \cos n\theta \sin\phi \, dr d\theta d\phi \\ d^*_{nm} = \dfrac{1}{N_0 N_{nm} N_{mk} \underline{\beta}_{mk}} \displaystyle\int_0^\pi \int_{-\pi}^\pi \int_0^{r_0} \psi(r, \theta, \phi) P_m^n(\cos\phi) j_m(\sqrt{\lambda_{mk}} r) r^2 \sin n\theta \sin\phi \, dr d\theta d\phi, \end{cases} \tag{3.97}$$

where N_0, N_{nm}, and N_{mk} are normal squares of $\{1, \cos\theta, \sin\theta, ..., \cos n\theta, \sin n\theta, ...\}$, $\{P_m^n(\cos\phi)\}$ and $\{j_m(\mu_k^{(m+1/2)}/\ r_0)\}$, respectively, given by

$$N_0 = \begin{cases} 2\pi, & n = 0 \\ \pi, & n > 0 \end{cases}, \tag{3.98}$$

$$N_{nm} = \int_0^{\pi} \left[P_m^n(\cos\phi)\right]^2 \sin\phi d\phi = \frac{(m+n)!}{(m-n)!}\frac{2}{2m+1}, \tag{3.99}$$

$$N_{mk} = \frac{\pi r_0^3}{4\mu_k^{(m+\frac{1}{2})}} \cdot \begin{cases} \left[J_{m+\frac{3}{2}}\left(\mu_k^{(m+\frac{1}{2})}\right)\right]^2, \\ \quad \text{1st type of boundary condition: } R(r_0) = 0 \\ \left[1 - \frac{m(m+1)}{\left(\mu_k^{(m+\frac{1}{2})}\right)^2}\right] J_{m+\frac{1}{2}}^2\left(\mu_k^{(m+\frac{1}{2})}\right), \\ \quad \text{2nd type of boundary condition: } R'(r_0) = 0 \\ \left[1 + \frac{(hr_0+m)(hr_0-m-1)}{\left(\mu_k^{(m+\frac{1}{2})}\right)^2}\right] J_{m+\frac{1}{2}}^2\left(\mu_k^{(m+\frac{1}{2})}\right), \\ \quad \text{3rd type of boundary condition: } R'(r_0) + hR(r_0) = 0. \end{cases} \tag{3.100}$$

Theorem 7. Let $W_\psi(r, \theta, \phi, t)$ be the solution of the $\psi-$ contribution problem (3.84). The solution of $\varphi-$ contribution problem

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = A^2 \Delta T(r, \theta, \phi, t) + B^2 \frac{\partial}{\partial t}\Delta T(r, \theta, \phi, t); \quad \Omega \times (0, +\infty) \\ \mathrm{L}\left(T, \frac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; \\ T(r, \theta, \phi, 0) = \varphi(r, \theta, \phi), \quad \frac{\partial T}{\partial t}(r, \theta, \phi, 0) = 0 \end{cases} \tag{3.101}$$

is

$$T(r, \theta, \phi, t) = \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right) W_\varphi(r, \theta, \phi) + B^2 W_{\lambda_{mk}\varphi}(r, \theta, \phi). \tag{3.102}$$

Proof. Following a similar approach as in solving the $\psi-$ contribution problem, we obtain the $T(r, \theta, \phi, t)$ that satisfies the equation and boundary conditions of (3.101),

$$\begin{aligned} T(r, \theta, \phi, t) = \sum_{n,m,k} e^{\alpha_{mk} t}[(a_{nmk}\cos\beta_{mk}t + b_{nmk}\underline{\sin}\beta_{mk}t)\cos n\theta + (c_{nmk}\cos\beta_{mk}t + d_{nmk}\underline{\sin}\beta_{mk}t)\sin n\theta] \\ \times P_m^n(\cos\phi) j_m(\lambda_{mk}^{1/2} r), \end{aligned} \tag{3.103}$$

where a_{nmk}, b_{nmk}, c_{nmk}, and d_{nmk} are determined by the initial conditions $T(r, \theta, \phi, 0) = \varphi(r, \theta, \phi)$ and $\partial T(r, \theta, \phi, 0)/\ \partial t = 0$,

$$\begin{cases} a_{nmk} = \frac{1}{N_0 N_{nm} N_{mk}} \int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{r_0} \varphi(r, \theta, \phi) P_m^n(\cos\phi) j_m(\sqrt{\lambda_{mk}} r) r^2 \cos n\theta \sin\phi dr d\theta d\phi \\ c_{nmk} = \frac{1}{N_0 N_{nm} N_{mk}} \int_0^{\pi}\int_{-\pi}^{\pi}\int_0^{r_0} \varphi(r, \theta, \phi) P_m^n(\cos\phi) j_m(\sqrt{\lambda_{mk}} r) r^2 \sin n\theta \sin\phi dr d\theta d\phi, \end{cases}$$

$$\begin{cases} \alpha_{mk} a_{nmk} + \underline{\beta}_{mk} b_{nmk} = 0, \quad \text{or } b_{nmk} = -\frac{\alpha_{mk}}{\underline{\beta}_{mk}} a_{nmk} \\ \alpha_{mk} c_{nmk} + \underline{\beta}_{mk} d_{nmk} = 0, \quad \text{or } d_{nmk} = -\frac{\alpha_{mk}}{\underline{\beta}_{mk}} c_{nmk}. \end{cases}$$

Based on the structure of $W_\psi(r,\theta,\phi,t)$ in (3.97), we have

$$\begin{cases} \dfrac{\partial W_\varphi(r,\theta,\phi,t)}{\partial t} = \displaystyle\sum_{n=0,m=0,k=1}^{\infty} \left(b^{**}_{nmk}\cos n\theta + d^{**}_{nmk}\sin n\theta\right)\left(\alpha_{mk}\underline{\sin}\underline{\beta}_{mk}t + \underline{\beta}_{mk}\cos\beta_{mk}t\right)e^{\alpha_{mk}t} \\ \qquad \times P^n_m(\cos\phi)j_m(\sqrt{\lambda_{mk}}r) \\ b^{**}_{nmk} = \dfrac{1}{N_0N_{nm}N_{mk}\underline{\beta}_{mk}}\displaystyle\int_0^\pi\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta,\phi)P^n_m(\cos\phi)j_m(\sqrt{\lambda_{mk}}r)r^2\cos n\theta\sin\phi drd\theta d\phi = \dfrac{a_{nmk}}{\underline{\beta}_{mk}} \\ d^{**}_{nm} = \dfrac{1}{N_0N_{nm}N_{mk}\underline{\beta}_{mk}}\displaystyle\int_0^\pi\int_{-\pi}^{\pi}\int_0^{r_0}\varphi(r,\theta,\phi)P^n_m(\cos\phi)j_m(\sqrt{\lambda_{mk}}r)r^2\sin n\theta\sin\phi drd\theta d\phi = \dfrac{c_{nmk}}{\underline{\beta}_{mk}}. \end{cases}$$

Also,

$$\begin{aligned} &\frac{1}{\tau_0}W_\varphi(r,\theta,\phi,t) + B^2W_{\lambda_{mk}\varphi}(r,\theta,\phi,t) \\ &\quad = \left(\frac{1}{\tau_0} + \lambda_{mk}B^2\right)\left(b^{**}_{nmk}\cos n\theta + d^{**}_{nmk}\sin n\theta\right)P^n_m(\cos\phi) \\ &\quad \times j_m(\sqrt{\lambda_{mk}}r)e^{\alpha_{mk}t}\underline{\sin}\beta_{mk}t. \end{aligned}$$

By adding $\partial W_\varphi(r,\theta,\phi,t)/\partial t$ and $(1/\tau_0)W_\varphi(r,\theta,\phi,t) + B^2W_{\lambda_{mk}\varphi}(r,\theta,\phi,t)$, we can readily prove that the result from Eq. (3.102) and the result obtained by separation of variables, Eq. (3.103), are the same.

Theorem 8. Let $W_\psi(r,\theta,\phi,t)$ be the solution of the $\psi-$ contribution problem (3.84). The solution of the $f-$ contribution problem

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T(r,\theta,\phi,t) + B^2\dfrac{\partial}{\partial t}\Delta T(r,\theta,\phi,t) + f(r,\theta,\phi,t); \quad \Omega\times(0,+\infty) \\ \mathrm{L}\left(T,\dfrac{\partial T}{\partial r}\right)\Big|_{r=r_0} = 0; \\ T(r,\theta,\phi,0) = 0, \quad \dfrac{\partial T}{\partial t}(r,\theta,\phi,0) = 0 \end{cases} \tag{3.104}$$

is

$$T(r,\theta,\phi,t) = \int_0^t W_{f_\tau}(r,\theta,\phi,t-\tau)d\tau, \tag{3.105}$$

where $f_\tau = f(r,\theta,\phi,\tau)$.

Again, Theorem 8 can be proved by following the same way as the proof of Theorem 2.

By the principle of superposition, the solution of the problem (3.83) is

$$\begin{aligned} T(r,\theta,\phi,t) &= W_\psi(r,\theta,\phi,t) + \left(\frac{1}{\tau_0} + \frac{\partial}{\partial t}\right)W_\varphi(r,\theta,\phi,t) \\ &\quad + B^2W_{\lambda_{mk}\varphi}(r,\theta,\phi) + \int_0^t W_{f_\tau}(r,\theta,\phi,t-\tau)d\tau, \end{aligned} \tag{3.106}$$

where $W_\psi(r,\theta,\phi,t)$ has been given in Eq. (3.97), $f_\tau = f(r,\theta,\phi,\tau)$. By substituting $W_\psi(r,\theta,\phi,t)$ into Eq. (3.105) we can have

$$\begin{aligned} T(r,\theta,\phi,t) &= \int_0^t W_{f_\tau}(r,\theta,\phi,t-\tau)d\tau \\ &= \int_0^t \iiint_\Omega G(r,r^*;\theta,\theta^*;\phi,\phi^*;t-\tau) \\ &\quad \times f(r^*,\theta^*,\phi^*,\tau)d\Omega d\tau, \end{aligned}$$

where

$$\begin{aligned} &G(r,r^*;\theta,\theta^*;\phi,\phi^*;t-\tau) \\ &\quad = \sum_{n=0,m=0,k=1}^{\infty}\frac{1}{N_0N_{nm}N_{mk}\underline{\beta}_{mk}}\cos n(\theta^*-\theta)\cdot P^n_m(\cos\phi) \\ &\quad \times P^n_m(\cos\phi^*)j_m(\sqrt{\lambda_{mk}}r)j_m(\sqrt{\lambda_{nm}}r^*)r^{*2}e^{\alpha_{nmk}(t-\tau)} \\ &\quad \times \underline{\sin}\beta_{nmk}(t-\tau). \end{aligned}$$

This is the *integral expression* of the solution of (3.104). The triple series $G(r,r^*;\theta,\theta^*;\phi,\phi^*;t-\tau)$ is called the *Green function of dual-phase-lagging heat conduction equation in a spherical domain*. When $f(r,\theta,\phi,\tau) = \delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$, in particular, the solution of (3.104) reduces to

$$T = G(r,r_0;\theta,\theta_0;\phi,\phi_0;t-t_0),$$

where $\mathbf{r} = (r,\theta,\phi)$, $\mathbf{r}_0 = (r_0,\theta_0,\phi_0)$. Thus, the Green function $G(r,r_0;\theta,\theta_0;\phi,\phi_0;t-t_0)$ is the solution due to the source term $\delta(\mathbf{r}-\mathbf{r}_0,t-t_0)$.

The results presented in Sec. III A also cover those for the hyperbolic heat conduction equations as the special case of DPL heat conduction equation at $B = 0$.

B. Solution structure theorems for Cauchy problems of DPL equations

If the heat conduction problems are defined in unbounded domain, then there exists no boundary condition. Such a problem is called the *initial-value problem*, or *Cauchy problem*. Solution structure theorems also exist for Cauchy problems:

Theorem 9. Let $W_\psi(M,t)$ be the solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\dfrac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ T(M,0)=0, \quad \dfrac{\partial T}{\partial t}(M,0)=\psi(M). \end{cases} \tag{3.107}$$

The solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\dfrac{\partial}{\partial t}\Delta T; \quad \Omega\times(0,+\infty) \\ T(M,0)=\varphi(M), \quad \dfrac{\partial T}{\partial t}(M,0)=0 \end{cases} \tag{3.108}$$

is

$$T(M,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi(M,t), \tag{3.109}$$

where Ω denotes one-, two-, or three-dimensional unbounded domain R^1, R^2, or R^3. M represents a point in Ω.

Proof. By its definition, $W_\varphi(M,t)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_\varphi}{\partial t}+\dfrac{\partial^2 W_\varphi}{\partial t^2}=A^2\Delta W_\varphi+B^2\dfrac{\partial}{\partial t}\Delta W_\varphi; \quad \Omega\times(0,+\infty) \\ W_\varphi(M,0)=0, \quad \dfrac{\partial W_\varphi}{\partial t}(M,0)=\varphi(M). \end{cases} \tag{3.110}$$

Substituting Eq. (3.109) into the equation of (3.108) and using the equation of (3.110) yields

$$\begin{aligned} &\frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}-A^2\Delta T-B^2\frac{\partial}{\partial t}\Delta T \\ &=\frac{1}{\tau_0}\frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\right]+\frac{\partial^2}{\partial t^2}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\right] \\ &\quad -A^2\Delta\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\right]-B^2\frac{\partial}{\partial t}\Delta\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\right] \\ &=\frac{1}{\tau_0}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right) \\ &\quad +\frac{\partial}{\partial t}\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right) \\ &\quad -B^2\Delta\left(\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-A^2\Delta W_\varphi-B^2\frac{\partial}{\partial t}\Delta W_\varphi\right)=0. \end{aligned}$$

Therefore, $T(M,t)$ in (3.109) satisfies the equation of (3.108).

We also have

$$\begin{aligned} T(M,0)&=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\bigg|_{t=0} \\ &=\frac{1}{\tau_0}W_\varphi(M,0)+\frac{\partial W_\varphi}{\partial t}(M,0)-B^2\Delta W_\varphi(M,0) \\ &=\varphi(M) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial T}{\partial t}&=\frac{\partial}{\partial t}\left[\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi\right] \\ &=\frac{1}{\tau_0}\frac{\partial W_\varphi}{\partial t}+\frac{\partial^2 W_\varphi}{\partial t^2}-B^2\frac{\partial}{\partial t}\Delta W_\varphi \\ &=A^2\Delta W_\varphi, \end{aligned}$$

such that

$$\frac{\partial T}{\partial t}\bigg|_{t=0}=A^2\Delta W_\varphi(M,0)=0.$$

Therefore, $T(M,t)$ in (3.109) also satisfies the two initial conditions of (3.108), thus verifying that it is the solution of the problem (3.108).

Theorem 10. Let $W_\psi(M,t)$ be the solution of (3.107). The solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\dfrac{\partial}{\partial t}\Delta T+f(M,t); \quad \Omega\times(0,+\infty) \\ T(M,0)=0, \quad \dfrac{\partial T}{\partial t}(M,0)=0 \end{cases} \tag{3.111}$$

is

$$T(M,t)=\int_0^t W_{f_\tau}(M,t-\tau)d\tau, \tag{3.112}$$

where $f_\tau=f(M,\tau)$.

Proof. By the definition of $W_\psi(M,t)$, the $W_{f_\tau}(M,t-\tau)$ satisfies

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial W_{f_\tau}}{\partial t}+\dfrac{\partial^2 W_{f_\tau}}{\partial t^2}=A^2\Delta W_{f_\tau}+B^2\dfrac{\partial}{\partial t}\Delta W_{f_\tau}; \quad \Omega\times(0,+\infty) \\ W_{f_\tau}(M,t-\tau)\big|_{t=\tau}=0, \quad \dfrac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)\bigg|_{t=\tau}=f(M,\tau). \end{cases} \tag{3.113}$$

Therefore, substituting Eq. (3.112) into (3.111) yields

$$\frac{1}{\tau_0}\frac{\partial T}{\partial t}+\frac{\partial^2 T}{\partial t^2}-A^2\Delta T-B^2\frac{\partial}{\partial t}\Delta T=\frac{1}{\tau_0}\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)d\tau+\frac{\partial^2}{\partial t^2}\int_0^t W_{f_\tau}(M,t-\tau)d\tau$$
$$-A^2\Delta\int_0^t W_{f_\tau}(M,t-\tau)d\tau-B^2\frac{\partial}{\partial t}\Delta\int_0^t W_{f_\tau}(M,t-\tau)d\tau$$
$$=\frac{1}{\tau_0}\left[\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}d\tau+W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]+\int_0^t\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}d\tau+\left.\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\right|_{\tau=t}$$
$$-A^2\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau-B^2\frac{\partial}{\partial t}\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau$$
$$=\frac{1}{\tau_0}\int_0^t\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}d\tau+\int_0^t\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}d\tau+\left.\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}\right|_{\tau=t}$$
$$-A^2\int_0^t\Delta W_{f_\tau}(M,t-\tau)d\tau-B^2\left[\int_0^t\frac{\partial}{\partial t}\Delta W_{f_\tau}(M,t-\tau)d\tau+\Delta W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]$$
$$=\int_0^t\left(\frac{1}{\tau_0}\frac{\partial W_{f_\tau}(M,t-\tau)}{\partial t}+\frac{\partial^2 W_{f_\tau}(M,t-\tau)}{\partial t^2}-A^2\Delta W_{f_\tau}(M,t-\tau)-B^2\frac{\partial}{\partial t}\Delta W_{f_\tau}(M,t-\tau)\right)d\tau$$
$$+f(M,t)=f(M,t).$$

Hence the $T(M,t)$ in Eq. (3.112) satisfies the equation of (3.111). Clearly, $T(M,t)$ in Eq. (3.112) also satisfies the initial condition $T(M,0)=0$. Also,

$$\frac{\partial T(M,0)}{\partial t}=\left.\frac{\partial}{\partial t}\int_0^t W_{f_\tau}(M,t-\tau)d\tau\right|_{t=0}$$
$$=\left.\left[\int_0^t\frac{\partial}{\partial t}W_{f_\tau}(M,t-\tau)d\tau+W_{f_\tau}(M,t-\tau)\big|_{\tau=t}\right]\right|_{t=0}$$
$$=0.$$

We have thus proved Theorem 10.

In summary, the solution of

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\dfrac{\partial}{\partial t}\Delta T+f(M,t); & \Omega\times(0,+\infty)\\ T(M,0)=\varphi(M), \quad \dfrac{\partial T}{\partial t}(M,0)=\psi(M)\end{cases}\tag{3.114}$$

can be written as, by the principle of superposition,

$$T(M,t)=\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\Delta\right)W_\varphi(M,t)+W_\psi(M,t)+\int_0^t W_{f_\tau}(M,t-\tau)d\tau,\tag{3.115}$$

where $W_\psi(M,t)$ is the solution of (3.107), and $f_\tau=f(M,\tau)$.

The $W_\psi(M,t)$ can be obtained by *integral transformation*: one-dimensional $W_\psi(M,t)$ can be obtained by using either the Fourier transformation with respect to the spatial variable x or the Laplace transformation with respect to the temporal variable t. Two- and three-dimensional $W_\psi(M,t)$ can be obtained by using multiple Fourier transformations. For example, assume $W_\psi(x,t)$ to be the solution of the one-dimensional problem:

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2 T}{\partial t^2}=A^2\Delta T+B^2\dfrac{\partial}{\partial t}\Delta T; & R^1\times(0,+\infty)\\ T(x,0)=0, \quad \dfrac{\partial T}{\partial t}(x,0)=\psi(x).\end{cases}\tag{3.116}$$

Its Laplace transformation reads

$$\frac{s}{\tau_0}\bar T(x,s)+s^2\bar T(x,s)-\psi(x)=A^2\frac{\partial^2}{\partial x^2}\bar T(x,s)+B^2 s\frac{\partial^2}{\partial x^2}\bar T(x,s)$$

or

$$\left(B^2 s+A^2\right)\frac{\partial^2}{\partial x^2}\bar T(x,s)-\left(s^2+\frac{s}{\tau_0}\right)\bar T(x,s)=-\psi(x),$$

so that

$$\frac{\partial^2}{\partial x^2}\bar T(x,s)-\frac{s^2+\frac{s}{\tau_0}}{B^2 s+A^2}\bar T(x,s)=-\frac{\psi(x)}{B^2 s+A^2},\tag{3.117}$$

where $\bar T(x,s)$ denotes the Laplace transformation of $T(x,t)$ with respect to t. The general solution of Eq. (3.117) is

$$\bar T(x,s)=c_1 e^{x\sqrt{a(s)}}+c_2 e^{-x\sqrt{a(s)}}-\frac{1}{2}\int\frac{b(s)}{\sqrt{a(s)}}e^{-(\xi-x)\sqrt{a(s)}}\times\psi(\xi)d\xi+\frac{1}{2}\int\frac{b(s)}{\sqrt{a(s)}}e^{(\xi-x)\sqrt{a(s)}}\psi(\xi)d\xi,$$

where

$$a(s)=\frac{s^2+\dfrac{s}{\tau_0}}{B^2 s+A^2}, b(s)=\frac{1}{B^2 s+A^2},$$

c_1 and c_2 are constants. Since $\bar{T}(x,s)$ must be bounded as $x \to \pm\infty$ so that $c_1 = c_2 = 0$. To ensure the convergence of the two integrals, let

$$\bar{T}(x,s) = -\frac{1}{2}\int_{+\infty}^{x} \frac{b(s)}{\sqrt{a(s)}} \mathrm{e}^{-(\xi-x)\sqrt{a(s)}}\psi(\xi)d\xi + \frac{1}{2}\int_{-\infty}^{x} \frac{b(s)}{\sqrt{a(s)}} \mathrm{e}^{(\xi-x)\sqrt{a(s)}}\psi(\xi)d\xi.$$

Therefore,

$$\begin{aligned} T(x,t) &= L^{-1}[\bar{T}(x,s)] \\ &= \frac{1}{2}\int_{x}^{+\infty} L^{-1}\left[\frac{b(s)}{\sqrt{a(s)}}\mathrm{e}^{-(\xi-x)\sqrt{a(s)}}\right]\psi(\xi)d\xi \\ &\quad + \frac{1}{2}\int_{-\infty}^{x} L^{-1}\left[\frac{b(s)}{\sqrt{a(s)}}\mathrm{e}^{(\xi-x)\sqrt{a(s)}}\right]\psi(\xi)d\xi. \end{aligned}$$

$T(x,t)$ can be obtained by the inverse Laplace transformation

$$\begin{aligned} L^{-1}\left[\frac{b(s)}{\sqrt{a(s)}}\mathrm{e}^{-(\xi-x)\sqrt{a(s)}}\right] &= \mathrm{e}^{-\frac{A^2}{B^2}t}\int_0^{t/B^2} J_0\left[\left(A^4 - \frac{B^2A^2}{\tau_0}\right)\right. \\ &\quad \times \left.\left(\frac{2t}{B^2}\tau - \tau^2\right)\right]\frac{1}{\sqrt{\pi\tau}} \\ &\quad \times \mathrm{e}^{\left(2A^2 - \frac{B^2}{\tau_0}\right)\tau - \frac{(\xi-x)^2}{4B^4\tau}}d\tau. \end{aligned}$$

The integrant in this equation can be obtained either analytically or numerically.

In the three-dimensional case, for another example, the $W_\psi(M,t)$ is the solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\Delta T + B^2\dfrac{\partial}{\partial t}\Delta T; \quad R^3 \times (0,+\infty) \\ T(M,0) = 0, \quad \dfrac{\partial T}{\partial t}(M,0) = \psi(M). \end{cases} \tag{3.118}$$

Its triple Fourier transformation yields

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\widehat{T}(\boldsymbol{\omega},t) + \left(\dfrac{1}{\tau_0} + B^2\boldsymbol{\omega}^2\right)\dfrac{\partial}{\partial t}\widehat{T}(\boldsymbol{\omega},t) + A^2\boldsymbol{\omega}^2\widehat{T}(\boldsymbol{\omega},t) = 0 \\ \widehat{T}(\boldsymbol{\omega},0) = 0, \quad \dfrac{\partial}{\partial t}\widehat{T}(\boldsymbol{\omega},0) = \widehat{\psi}(\boldsymbol{\omega}), \end{cases} \tag{3.119}$$

where $\widehat{T}(\boldsymbol{\omega},t)$ denotes the Fourier transformation of $T(x,y,z,t)$ with respect to the three spatial variables; $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$. Let $a(\boldsymbol{\omega})\pm\mathbf{i}\beta(\boldsymbol{\omega})$ be the characteristic roots of the equation of (3.119). Then

$$\bar{u}(\boldsymbol{\omega},t) = \mathrm{e}^{\alpha(\boldsymbol{\omega})t}[A(\boldsymbol{\omega})\cos\beta(\boldsymbol{\omega})t + B(\boldsymbol{\omega})\underline{\sin}\beta(\boldsymbol{\omega})t],$$

where $A(\boldsymbol{\omega})$ and $B(\boldsymbol{\omega})$ are functions of $\boldsymbol{\omega}$ to be determined. Applying the initial conditions yields

$$A(\boldsymbol{\omega})=0, B(\boldsymbol{\omega})=\frac{\psi(\boldsymbol{\omega})}{\beta(\boldsymbol{\omega})},$$

Thus,

$$\bar{u}(\boldsymbol{\omega},t) = \frac{\bar{\psi}(\boldsymbol{\omega})}{\underline{\beta}(\boldsymbol{\omega})}e^{\alpha(\boldsymbol{\omega})t}\underline{\sin}\beta(\boldsymbol{\omega})t.$$

The solution of (3.118) can be obtained through the inverse Fourier transformation

$$u(M,t) = \frac{1}{(2\pi)^3}\iiint_{R^3}\bar{u}(\boldsymbol{\omega},t)e^{\mathbf{i}(omega_1 x + \omega_2 y + \omega_3 z)}d\omega_1 d\omega_2 d\omega_3.$$

Note that in the problem of (3.107), $A^2 = \alpha/\tau_q$ and $B^2 = \alpha\tau_T/\tau_q$. For ordinary materials α, τ_q and τ_T are very small, so that $B^2 \ll A^2$ and $B^2 \ll 1$. At these points, we can obtain an approximate analytical solution of (3.107) by using the *perturbation method* with respect to B^2 and making use of the solution of (3.107) at $B^2 = 0$. In the following, we take the one-dimensional problem in Cartesian coordinates as an example to illustrate this method.

For the case of $B^2 = \varepsilon \ll 1$, one-dimensional Cauchy problem of DPL conduction reads

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\dfrac{\partial^2 T}{\partial x^2} + \varepsilon\dfrac{\partial^3 T}{\partial t\partial x^2}; \quad R^1 \times (0,+\infty) \\ T(x,0) = 0, \quad \dfrac{\partial T}{\partial t}(x,0) = \psi(x). \end{cases} \tag{3.120}$$

Solving it by perturbation method relies on the solution of

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t} + \dfrac{\partial^2 T}{\partial t^2} = A^2\dfrac{\partial^2 T}{\partial x^2}; \quad R^1 \times (0,+\infty) \\ T(x,0) = 0, \quad \dfrac{\partial T}{\partial t}(x,0) = \psi(x). \end{cases} \tag{3.121}$$

It is a hyperbolic heat conduction problem, whose solution can be obtained by, for example, Riemann method[16]

$$T_0(x,t) = \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\int_{-At}^{At} I_0\left(\frac{1}{2A\tau_0}\sqrt{(At)^2 - u^2}\right)\psi(u+x)du, \tag{3.122}$$

where $I_0(x)$ is the modified Bessel function of the first kind and zeroth order. An approximate analytical solution of (3.120) can be obtained by correcting $T(x,t)$ in (3.122) via a polynomial with respect to ε. In particular, when $\psi(x)$ is a polynomial of x, such as

$$\psi(x) = P_N(x) = \sum_{n=0}^{N} a_n x^n.$$

The perturbation method can lead to the exact solution of (3.120). Since elementary functions can normally be approximated by Taylor polynomials, we focus our discussion on the solution of (3.120) with a polynomial $\psi(x)$.

With $\psi(x) = P_N(x) = \sum_{n=0}^{N} a_n x^n$, the solution of (3.121) takes the form of

$$T_0(x,t) = \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\int_{-At}^{At} I_0\left(\frac{1}{2A\tau_0}\sqrt{(At)^2-u^2}\right) \times \left[\sum_{n=0}^{N} a_n(u+x)^n\right]du$$
$$= \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\int_{-At}^{At} I_0\left(\frac{1}{2A\tau_0}\sqrt{(At)^2-u^2}\right) \times \left[\sum_{n=0}^{N} a_n\left(\sum_{i=0}^{n} C_n^i u^i x^{n-i}\right)\right]du. \quad (3.123)$$

By defining

$$G_i(t) = \int_{-At}^{At} I_0\left(\frac{1}{2A\tau_0}\sqrt{(At)^2-u^2}\right)u^i du, \quad i = 0, 1, ..., n,$$

$T(x,t)$ can be rewritten as

$$T_0(x,t) = \tau_0 P_N(x)\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right) + \frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left(\sum_{i=1}^{[N/2]}\frac{P_N^{(2i)}(x)}{(2i)!}G_{2i}(t)\right), \quad (3.124)$$

in which the properties of $G_i(t)$ that when i is odd, $G_i(t) = 0$, and

$$G_0(t) = \int_{-At}^{+At} I_0\left(b\sqrt{(At)^2-u^2}\right)du = 2A\tau_0 e^{\frac{t}{2\tau_0}}\left(1-e^{-\frac{t}{\tau_0}}\right)$$

have been used. $[N/2]$ denotes the maximum positive integer not larger than $N/2$. Equation (3.124) shows that the solution is a sum of $[N/2]+1$ terms. All terms are in the variable-separable form.

Let the solution of (3.120) with $\psi(x) = P_N(x) = \sum_{n=0}^{N} a_n x^n$ be

$$T(x,t) = T_0(x,t) + T_1(x,t)\varepsilon + T_2(x,t)\varepsilon^2 + \cdots + T_n(x,t)\varepsilon^n + \cdots, \quad (3.125)$$

where $T_n(x,t)$ are functions to be determined. Substituting Eq. (3.125) into (3.120) and comparing the coefficients of $\varepsilon^n (n = 0,\ 1,\ 2, ...)$ terms yields the problems of $T_n(x,t)$, $(n = 0,\ 1,\ 2, ...)$:

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T_0}{\partial t} + \dfrac{\partial^2 T_0}{\partial t^2} = A^2\dfrac{\partial^2 T_0}{\partial x^2}; & R^1 \times (0, +\infty) \\ T_0(x,0) = 0, \quad \dfrac{\partial T_0}{\partial t}(x,0) = P_N(x), \end{cases}$$

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T_1}{\partial t} + \dfrac{\partial^2 T_1}{\partial t^2} = A^2\dfrac{\partial^2 T_1}{\partial x^2} + \dfrac{\partial^3 T_0}{\partial t \partial x^2}; & R^1 \times (0, +\infty) \\ T_1(x,0) = 0, \quad \dfrac{\partial T_1}{\partial t}(x,0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial T_2}{\partial t} + \dfrac{\partial^2 T_2}{\partial t^2} = A^2\dfrac{\partial^2 T_2}{\partial x^2} + \dfrac{\partial^3 T_1}{\partial t \partial x^2}; & R^1 \times (0, +\infty) \\ T_2(x,0) = 0, \quad \dfrac{\partial T_2}{\partial t}(x,0) = 0 \end{cases}$$

$$\cdots \quad \cdots$$

$T_0(x,t)$ is given by Eq. (3.124), which is a N-th polynomial of x

$T_1(x,t)$ is the solution of $f-$contribution hyperbolic heat conduction problem with $f(x,t)$ being $\partial^3 T_0/\partial t \partial x^2$:[16]

$$T_1(x,t) = \int_0^t W_{\frac{\partial^3 T_0}{\partial t \partial x^2}\big|_{t=\tau}}(x, t-\tau)d\tau = \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau\int_{x-A(t-\tau)}^{x+A(t-\tau)} \times I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right)\frac{\partial^3 T_0(\xi,\tau)}{\partial\tau\partial\xi^2}d\xi,$$

which is a $(N-2)$-th polynomial of x. Similarly, $T_n(x,t)$ is a $(N-2n)$-th polynomial of x and $T_n(x,t) = 0$ at $n = [N/2] + 1, [N/2]+2, ...$ Therefore, the analytical solution of (3.120) with $\psi(x) = P_N(x) = \sum_{n=0}^{N} a_n x^n$ can be expressed as

$$T(x,t) = T_0(x,t) + T_1(x,t)\varepsilon + T_2(x,t)\varepsilon^2 + \cdots + T_{[N/2]}(x,t)\varepsilon^{[N/2]}. \quad (3.126)$$

Define an operator

$$S = \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau\int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right) \times \frac{\partial^3}{\partial t\partial x^2}\bigg|_{\substack{x=\xi \\ t=\tau}} d\xi,$$

so that $T_1 = S(T_0)$, $T_2 = S(T_1) = S^2(T_0), ..., T_{[N/2]} = S^{[N/2]}(T_0)$. The solution of (3.120) can be further reduced to

$$T(x,t) = T_0 + S(T_0)\varepsilon + S^2(T_0)\varepsilon^2 + \cdots + S^{[N/2]}(T_0)\varepsilon^{[N/2]}. \quad (3.127)$$

Therefore, the task of finding the solution $T(x,t)$ of problem (3.120) reduces to that of applying the operator S to $T_0(x,t)$. $T_0(x,t)$ has the same form for both even N $(N = 2m)$ and odd N $(N = 2m+1)$. Therefore, $T_1(x,t)$ is also the same for both $N = 2m$ and $N = 2m+1$,

$$\begin{aligned} T_1(x,t) &= S(T_0) \\ &= \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau\int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-(x-\xi)^2}\right) \\ &\quad \times\left[P''_N(\xi)e^{-\frac{\tau}{\tau_0}} + \frac{1}{2A}\sum_{i=1}^{m-1}\frac{P_N^{(2i+2)}(\xi)}{(2i)!}g_{2i}(\tau)\right]d\xi \\ &= \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau\int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\cdot P''_N(\xi)e^{-\frac{\tau}{\tau_0}}d\xi \\ &\quad + \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau\int_{x-A(t-\tau)}^{x+A(t-\tau)} I_0\cdot\left[\frac{1}{2A}\sum_{i=1}^{m-1}\frac{P_N^{(2i+2)}(\xi)}{(2i)!}g_{2i}(\tau)\right]d\xi \\ &= \tau_0 P''_N(x)\left[\tau_0 + (\tau_0+t)e^{-\frac{t}{\tau_0}}\right] \\ &\quad + \frac{1}{2A}\sum_{i=1}^{m-1}\frac{P_N^{(2i+2)}(x)}{(2i)!}\int_0^t e^{-\frac{t+\tau}{2\tau_0}}G_{2i}(t-\tau)d\tau \\ &\quad + \frac{1}{2A}\sum_{i=1}^{m-1}\frac{P_N^{(2i+2)}(x)}{(2i)!}\int_0^t \tau_0 g_{2i}(\tau)\left(1-e^{-\frac{t-\tau}{2\tau_0}}\right)d\tau \\ &\quad + \frac{1}{4A^2}\sum_{\substack{i,j=1 \\ i+j\le m-1}}^{m-2}\frac{P_N^{(2i+2j+2)}(x)}{(2i)!(2j)!}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}G_{2i}(t-\tau)g_{2j}(\tau)d\tau, \end{aligned} \quad (3.128)$$

where $g_{2i}(\tau) = d\left[e^{-\tau/(2\tau_0)}G_{2i}(\tau)\right]/d\tau$. $T_2(x,t)$, $T_3(x,t)$, $\ldots$, $T_{[N/2]}(x,t)$ can be obtained by reapplying the operator S to u_0. Otherwise, $T_{[N/2]}(x,t) = S^{[N/2]}(u_0)$ can be also determined by the following approach.

Note that

$$S^{[N/2]}(T_0) = P_N^{2[N/2]}(x)q(t), \tag{3.129}$$

where $q(t)$ is a function to be determined. By introducing the operator

$$\Lambda = \frac{1}{2A}\int_0^t e^{-\frac{t-\tau}{2\tau_0}}d\tau \int_{-A(t-\tau)}^{A(t-\tau)} I_0\left(b\sqrt{A^2(t-\tau)^2-u^2}\right)\frac{d}{dt}\bigg|_{t=\tau}du$$
$$= \tau_0\int_0^t e^{-\frac{t-\tau}{\tau_0}}\frac{d}{dt}\bigg|_{t=\tau}d\tau,$$

we have

$$q(t) = \Lambda^{[N/2]}\left[\tau_0\left(1-e^{-\frac{t}{\tau_0}}\right)\right]$$
$$= \tau_0\int_0^t\left(1-e^{-\frac{t-\tau}{\tau_0}}\right)\frac{\tau^{[N/2]-1}}{([N/2]-1)!}e^{-\frac{\tau}{\tau_0}}d\tau$$
$$= \tau_0^{[N/2]}\left[\tau_0-\left(\tau_0+t+\frac{t^2}{2!\tau_0}+\cdots+\frac{\tau^{[N/2]}}{[N/2]!\tau_0^{[N/2]-1}}\right)e^{-\frac{t}{\tau_0}}\right].$$

In applications, the order of polynomial $P_N(x)$ is normally not larger than 5 so that $[N/2] \le 2$. Therefore, the determination of $T(x,t)$ by applying the operator S is not as complicated as it looks.

For $N = 5$, for example, $T_0(x,t)$ is given by Eq. (3.124):

$$T_0(x,t) = \tau_0 P_5(x)\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)$$
$$+\frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\left(\frac{P''_5(x)}{2}G_2(t)+\frac{P_5^{(4)}(x)}{4!}G_4(t)\right).$$

$T_1 = S(T_0)$ can be obtained by Eq. (3.128), whose last term vanishes:

$$T_1(x,t) = S(T_0)$$
$$= \tau_0 P''_5(x)\left[\tau_0+(\tau_0+t)e^{-\frac{t}{\tau_0}}\right]+\frac{1}{2A}\frac{P_5^{(4)}(x)}{2}$$
$$\times\int_0^t e^{-\frac{t+\tau}{2\tau_0}}G_2(t-\tau)d\tau+\frac{1}{2A}\frac{P_5^{(4)}(x)}{2}$$
$$\times\int_0^t \tau_0 g_2(\tau)\left(1-e^{-\frac{t-\tau}{2\tau_0}}\right)d\tau.$$

$T_2 = S^2(T_0)$ can be directly written out by Eq. (3.129):

$$T_2 = S^2(T_0) = \tau_0^2 P_5^{(4)}(x)\left[\tau_0-\left(\tau_0+t+\frac{t^2}{2\tau_0}\right)e^{-\frac{t}{\tau_0}}\right].$$

Finally, $T(x,t) = T_0 + S(T_0)\varepsilon + S^2(T_0)\varepsilon^2$. The result for $N = 4$ has the same form as that for $N = 5$ by only replacing $P_5(x)$ with $P_4(x)$.

For $N = 3$ $(P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3)$,

$$T_0(x,t) = \tau_0 P_3(x)\left(1-\mathrm{e}^{-\frac{t}{\tau_0}}\right)+\frac{1}{2A}\mathrm{e}^{-\frac{t}{2\tau_0}}\frac{P''_3(x)}{2}G_2(t).$$

$T_1 = S(T_0)$ can be obtained by Eq. (3.129):

$$T_1 = S(T_0) = \tau_0 P''_3(x)\left[\tau_0-(\tau_0+t)e^{-\frac{t}{\tau_0}}\right]$$
$$= \tau_0(2a_2+6a_3x)\left[\tau_0-(\tau_0+t)e^{-\frac{t}{\tau_0}}\right],$$

so that $T(x,t) = T_0 + S(T_0)\varepsilon$. Similarly, the result for $N = 2$ can be readily obtained by using $a_3 = 0$. It shows that the effect of $\partial^3T/(\partial t\partial x^2)$-term is x-independent for the case of $P_2(x)$ and increases to $2a_2\tau_0^2$ as $t \to \infty$.

For $N = 1$ $(P_1(x) = a_0 + a_1x)$,

$$T(x,t) = T_0(x,t) = \tau_0(a_0+a_1x)\left(1-e^{-\frac{t}{\tau_0}}\right).$$

This shows that the $\partial^3T/(\partial t\partial x^2)$-term has no effect on the solution of an initial value of type $P_1(x)$.

By the solution structure Theorems 9 and 10 [Eqs. (3.109) and (3.112)], the solution of the problem

$$\begin{cases}\dfrac{1}{\tau_0}\dfrac{\partial T}{\partial t}+\dfrac{\partial^2T}{\partial t^2}=A^2\dfrac{\partial^2T}{\partial x^2}+B^2\dfrac{\partial^3T}{\partial t\partial x^2}+P_{k_t}(x,t); & R^1\times(0,+\infty)\\ T(x,0)=P_m(x), \quad \dfrac{\partial T}{\partial t}(x,0)=P_n(x)\end{cases} \tag{3.130}$$

is

$$T(x,t) = \left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}-B^2\frac{\partial^2}{\partial x^2}\right)W_{P_m}(x,t)+W_{P_n}(x,t)$$
$$+\int_0^t W_{P_{k_\tau}}(x,t-\tau)d\tau, \tag{3.131}$$

where $P_m(x)$, $P_n(x)$, and $P_{k_t}(x,t)$ are m-, n-, and k-th polynomials of x and $P_{k_t}(x,t) = \sum_{i=0}^k a_i(t)x^i$. $P_{k_\tau} = P_{k_t}(x,\tau)$.

The perturbation method can be similarly applied for two- and three-dimensional Cauchy problems, by taking use of the solutions for two- and three-dimensional Cauchy problems of hyperbolic heat conduction.

IV. BIOHEAT TRANSPORT IN SKIN TISSUE AND DURING MAGNETIC HYPERTHERMIA

The complicated microscopic physics in biological tissues leads to non-Fourier behavior of heat conduction. This has been experimentally observed and has been attracting increasingly more attention.[21,25] Our rigorous development of the macroscopic heat transport model also supports a dual-phase-lagging bio-heat model in biological tissues. Here we study the bioheat transport in skin and during magnetic hyperthermia by using solution structure theorems developed in Sec. III to solve DPL bioheat model. The solutions of the corresponding Pennes model and thermal wave model (TWMBT) are also analytically obtained as special cases of DPL model at $\tau_q = \tau_T = 0$ and $\tau_q > \tau_T = 0$, respectively, and compared with the DPL bioheat model.

A. Skin bioheat transport

Several thermal therapies and physiological functions involve heat transport in skin tissue. Successful thermal treatments and deep understanding of physiological processes require an accurate prediction of the response of skin tissue to external thermal condition. The solution structure theorems discussed in Sec. III is capable of facilitating this prediction considerably.

For the sake of illustration, consider the same problem as described by Xu *et al.*:[21] the skin is initially in equilibrium with environmental air where natural convection boundary condition is applicable ($T_e = 25\,^\circ\mathrm{C}$, $h_0 = 7\mathrm{W}/(\mathrm{m}^2\cdot\mathrm{K})$). At $t = 0$, the skin surface begins to contact with a hot source of constant temperature $T_1 = 100\,^\circ\mathrm{C}$ for a period of 15 s; after removing the hot source, the skin is then cooled by a cold source at constant temperature $T_2 = 0\,^\circ\mathrm{C}$ for another period of 30 s. In Ref. 21, the skin is modeled as a four-layer structure so that different layers can have different thermophysical properties. Temperature profiles are obtained by employing numerical approaches to solve three kinds of continuum models: Pennes model, thermal wave model, and DPL models of different orders. While predictions of the three different-order DPL models are shown to be close from each other, they are significantly distinct from those of Pennes model and thermal wave model. In this section, we solve Pennes model, thermal wave model and first-order DPL model analytically by regarding the skin as a single homogeneous layer and assuming constant properties in the skin. Temperature profiles are obtained under two sets of skin properties, and compared with the numerical predictions for multi-layer structure to show the effectiveness of the one-layer approximation.

The governing equations, boundary conditions, and initial conditions are given by:

Pennes model:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\rho_b c_b \omega_b}{\rho c}[T_a - T(x,t)] + \frac{Q_m}{\rho c}; \tag{4.1}$$

Thermal wave model:

$$\left(\frac{1}{\tau_q} + \frac{\rho_b c_b \omega_b}{\rho c}\right)\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = \frac{\alpha}{\tau_q}\frac{\partial^2 T}{\partial x^2} + \frac{1}{\tau_q}\frac{\rho_b c_b \omega_b}{\rho c} \times [T_a - T(x,t)] + \frac{1}{\tau_q}\frac{Q_m}{\rho c}; \tag{4.2}$$

DPL model:

$$\left(\frac{1}{\tau_q} + \frac{\rho_b c_b \omega_b}{\rho c}\right)\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = \frac{\alpha}{\tau_q}\frac{\partial^2 T}{\partial x^2} + \alpha\frac{\tau_T}{\tau_q}\frac{\partial}{\partial t}\left(\frac{\partial^2 T}{\partial x^2}\right) + \frac{1}{\tau_q}\frac{\rho_b c_b \omega_b}{\rho c}[T_a - T(x,t)] + \frac{1}{\tau_q}\frac{Q_m}{\rho c}; \tag{4.3}$$

Boundary conditions:

$$\begin{aligned} &T(0,t) = T_1 \text{ (heating process)};\\ &T(0,t) = T_2 \text{ (cooling process)};\\ &\left.\frac{\partial T}{\partial x}\right|_{x=l} = 0; \end{aligned} \tag{4.4}$$

Initial conditions:

$$\begin{aligned} &T(x,0) = T_0(x): \begin{cases} \alpha\dfrac{d^2T_0}{dx^2} + \dfrac{\rho_b c_b \omega_b}{\rho c}[T_a - T_0(x)] + \dfrac{Q_m}{\rho c} = 0 \\ -k\left.\dfrac{dT_0}{dx}\right|_{x=0} = h_0[T_e - T_0(0)] \\ T_0(l) = T_c \end{cases}\\ &\left.\frac{\partial T}{\partial t}\right|_{t=0} = 0. \end{aligned} \tag{4.5}$$

In Eqs. (4.1)–(4.5), T is the temperature, x denotes the distance to the skin surface. α, k, ρ, and c are thermal diffusivity, thermal conductivity, density, and specific heat of tissue, respectively; ρ_b, c_b, andω_b are density, specific heat, and perfusion rate of blood, respectively; T_a is temperature of the arterial blood supply; Q_m is the metabolic heat generation; τ_q andτ_T are the phase lags of heat flux and temperature gradient of tissue, respectively. $T_0(x)$ denotes the initial temperature distribution in the tissue, which satisfies the steady-state governing equations of T, representing the balance among heat conduction, blood perfusion, metabolic heat generation, and heat loss through natural convection. h_0 and T_e are convective heat transfer coefficient for natural convection and environment temperature, respectively. T_c and l are the deep body temperature and skin depth of interest. l should be large enough so that the adiabatic boundary condition at $x = l$ is applicable.

Solving of the continuum models requires reliable data about the macroscale properties of the continuum. Their experimental data at different layers in skin are listed in Table III.[21] Based on these data, Xu *et al.*[21] also recommended the overall skin properties when the skin is treated as single-layer structure (Set 1 in Table III). We will solve the three models by using both the suggested property values in Xu *et al.*[21] and another set of values (Set 2 in Table III). Detailed solution procedure is demonstrated in the following for the DPL model (4.3), while solutions of the other two can be similarly obtained at $\tau_q = \tau_T = 0$ or $\tau_q > \tau_T = 0$.

TABLE III. Thermophysical properties of different layers in skin and blood.

Parameters	Stratum corneum (0.01 mm)	Epidermis (0.08 mm)	Dermis (1.5 mm)	Fat (4.4 mm)	Overall Set 1 [21]	Overall Set 2
Thermal conductivity, k (W/m K)	0.235	0.235	0.445	0.185	0.235	0.35
Density, ρ (kg/m^3)	1500	1190	1116	971	1190	1100
Specific heat, c (J/kg K)	3600	3600	3300	2700	3600	3200
Metabolic heat generation, Q_m(W/m^3)	368.1	368.1	368.1	368.3	368.1	368.1
Blood density, ρ_b (kg/ m^3)			1060			
Blood specific heat, c_b (J/kg K)			3770			
Arterial blood temperature, T_a(°C)			37			
Core body temperature, T_c(°C)			37			
Blood perfusion rate, ω_b (s^{-1})			0.001			

(*i*) *Heating process:* $0 < t < 15$ s.

By solving (4.5), we can obtain the initial temperature distribution $T_0(x)$ *at* $t = 0$s:

$$T_0(x) = A_0 + c_1 e^{B_0 x} + c_2 e^{-B_0 x},$$

where

$$A_0 = T_b + \frac{Q_m}{\rho_b c_b \omega_b}, \quad B_0 = \sqrt{\frac{\rho_b c_b \omega_b}{k}},$$

$$c_1 = \frac{(h_0 + kB_0)(T_c - A_0) - h_0 e^{-B_0 l}(T_e - A_0)}{e^{B_0 l}(h_0 + kB_0) - e^{-B_0 l}(h_0 - kB_0)},$$

$$c_2 = \frac{h_0 e^{B_0 l}(T_e - A_0) - (h_0 - kB_0)(T_c - A_0)}{e^{B_0 l}(h_0 + kB_0) - e^{-B_0 l}(h_0 - kB_0)}. \quad (4.6)$$

To remove the $T(x,t)-$ term in the source term of (4.3), consider a function transformation,

$$\Theta(x,t) = [T(x,t) - T_0(x)]e^{\frac{\rho_b c_b \omega_b}{\rho c}x},$$

so that we obtain the governing equation regarding $\Theta(x, t)$,

$$\begin{cases} \left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\right)\frac{\partial \Theta}{\partial t} + \frac{\partial^2 \Theta}{\partial t^2} = \alpha\left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\frac{\tau_T}{\tau_q}\right)\frac{\partial^2 \Theta}{\partial x^2} + \alpha\frac{\tau_T}{\tau_q}\frac{\partial}{\partial t}\left(\frac{\partial^2 \Theta}{\partial x^2}\right) \\ \Theta(0,t) = [T_1 - T_0(0)]e^{\frac{\rho_b c_b \omega_b}{\rho c}t} = g_1(t), \quad \Theta(l,t) = 0 \\ \Theta(x,0) = 0, \quad \left.\frac{\partial \Theta}{\partial t}\right|_{t=0} = 0. \end{cases} \quad (4.7)$$

To have a homogeneous boundary condition, let

$$\Theta(x,t) = \Theta_1(x,t) + \Theta_2(x,t), \text{ where } \Theta_2(x,t) = \left(1 - \frac{x}{l}\right)g_1(t).$$

Substituting it into Eq. (4.7), we have the mixed initial-boundary problem for $\Theta_1(x,t)$:

$$\begin{cases} \frac{1}{\tau_0}\frac{\partial \Theta_1}{\partial t} + \frac{\partial^2 \Theta_1}{\partial t^2} = A^2\frac{\partial^2 \Theta_1}{\partial x^2} + B^2\frac{\partial}{\partial t}\left(\frac{\partial^2 \Theta_1}{\partial x^2}\right) + f(x,t) \\ \Theta_1(0,t) = \Theta_1(l,t) = 0 \\ \Theta_1(x,0) = -\Theta_2(x,0) = \left(\frac{x}{l} - 1\right)g_1(0) = \varphi(x) \\ \left.\frac{\partial \Theta_1}{\partial t}\right|_{t=0} = -\left.\frac{\partial \Theta_2}{\partial t}\right|_{t=0} = \left(\frac{x}{l} - 1\right)g'_1(0) = \psi(x), \end{cases} \quad (4.8)$$

where

$$\frac{1}{\tau_0} = \left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\right), \quad A^2 = \alpha\left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\frac{\tau_T}{\tau_q}\right),$$

$$B^2 = \alpha\frac{\tau_T}{\tau_q}, \quad f(x,t) = \left(\frac{x}{l} - 1\right)\left[\frac{1}{\tau_0}g'_1(t) + g''_1(t)\right]. \quad (4.9)$$

We can readily solve the problem (4.8) by following the procedure in Sec. III.

By Eq. (3.15) and Table II, the solution of $\psi(x)-$ contribution problem at $f(x,t) = \varphi(x) = 0$ can be readily written

$$\begin{cases} W_\psi(x,t) = \sum_{n=1}^{\infty} b_n e^{\alpha_n t}\underline{\sin}\beta_n t \cdot \sin\frac{n\pi x}{l} \\ b_n = \frac{2}{l\underline{\beta}_n}\int_0^l g'_1(0)\left(\frac{x}{l} - 1\right)\sin\frac{n\pi x}{l}dx, \end{cases} \quad (4.10)$$

where

$$\alpha_n = -\frac{1}{2}\left[\frac{1}{\tau_0} + \left(\frac{n\pi}{l}\right)^2 B^2\right],$$

$$\beta_n = \frac{1}{2}\sqrt{4\left(\frac{n\pi}{l}\right)^2 A^2 - \left[\frac{1}{\tau_0} + \left(\frac{n\pi}{l}\right)^2 B^2\right]^2},$$

$1/\tau_0$, A^2, and B^2 are given in (4.9).

According to Theorem 1, the solution of the $\varphi(x)$-contribution problem is

$$\begin{aligned}
&\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(x,t)+B^2W_{\left(\frac{n\pi}{l}\right)^2\varphi}(x,t)\\
&=\frac{1}{\tau_0}\sum_{n=1}^{\infty}b_n^*e^{\alpha_n t}\underline{\sin}\beta_n t\cdot\sin\frac{n\pi x}{l}\\
&\quad+\sum_{n=1}^{\infty}b_n^*e^{\alpha_n t}\left(\alpha_n\underline{\sin}\beta_n t+\underline{\beta}_n\cos\beta_n t\right)\sin\frac{n\pi x}{l}\\
&\quad+B^2\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^2 b_n^*e^{\alpha_n t}\underline{\sin}\beta_n t\cdot\sin\frac{n\pi x}{l}\\
&=\sum_{n=1}^{\infty}\left[\frac{1}{\tau_0}+\alpha_n+B^2\left(\frac{n\pi}{l}\right)^2\right]b_n^*e^{\alpha_n t}\underline{\sin}\beta_n t\cdot\sin\frac{n\pi x}{l}\\
&\quad+\sum_{n=1}^{\infty}\underline{\beta}_n b_n^*e^{\alpha_n t}\cos\beta_n t\cdot\sin\frac{n\pi x}{l}\\
&=\sum_{n=1}^{\infty}\left(-\alpha_n\underline{\sin}\beta_n t+\underline{\beta}_n\cos\beta_n t\right)b_n^*e^{\alpha_n t}\sin\frac{n\pi x}{l},
\end{aligned} \tag{4.11}$$

where

$$b_n^*=\frac{2}{l\underline{\beta}_n}\int_0^l g_1(0)\left(\frac{x}{l}-1\right)\sin\frac{n\pi x}{l}dx.$$

According to Theorem 2, the solution of the *f(x, t)* contribution problem reads

$$\int_0^t W_{f_\tau(x,t-\tau)}d\tau=\int_0^l\int_0^t G(x,\xi,t-\tau)f(\xi,\tau)d\tau d\xi, \tag{4.12}$$

where

$$G(x,\xi,t-\tau)=\sum_{n=1}^{\infty}\frac{2}{l\underline{\beta}_n}\mathrm{e}^{\alpha_n(t-\tau)}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\underline{\sin}\beta_n(t-\tau).$$

Therefore, the solution of the problem (4.8) has the form of

$$\Theta_1(x,t)=W_{\psi}(x,t)+\left(\frac{1}{\tau_0}+\frac{\partial}{\partial t}\right)W_{\varphi}(x,t)+B^2W_{\left(\frac{n\pi}{l}\right)^2\varphi}(x,t)+\int_0^t W_{f_\tau(x,t-\tau)}d\tau. \tag{4.13}$$

We can further obtain $\Theta(\mathrm{x,t})$ and $T(x,t)$ by $\Theta(\mathrm{x,t})=\Theta_1(x,t)+\Theta_2(x,t)$ and $T(x,t)=\Theta(\mathrm{x,t})\mathrm{e}^{-\rho_b c_b\omega_b x/(\rho c)}+T_0(x)$, respectively.

(*ii*) *Cooling process:* $15<t<45$ s.

The initial temperature for the cooling process should be the value of $T(x,t)$ *at* $t=15$ s, rather than the $T_0(x)$ satisfying (4.5). Thus, the problem of $\Theta^*(x,t)=[T(x,t)-T_0^*(x)]\mathrm{e}^{-\rho_b c_b\omega_b x/(\rho c)}$ has the form of:

$$\begin{cases}
\dfrac{1}{\tau_0}\dfrac{\partial\Theta^*}{\partial t}+\dfrac{\partial^2\Theta^*}{\partial t^2}\\
\quad=A^2\dfrac{\partial^2\Theta^*}{\partial x^2}+B^2\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2\Theta^*}{\partial x^2}\right)+\dfrac{1}{\tau_q}\left\{\alpha\dfrac{d^2T_0^*}{dx^2}+\dfrac{\rho_b c_b\omega_b}{\rho c}\left[T_a-T_0^*(x)\right]+\dfrac{Q_m}{\rho c}\right\}\mathrm{e}^{\frac{\rho_b c_b\omega_b}{\rho c}t}\\
\Theta^*(0,t)=\left[T_2-T_0^*(0)\right]\mathrm{e}^{\frac{\rho_b c_b\omega_b}{\rho c}t}=g_2(t),\quad\Theta^*(l,t)=0\\
\Theta^*(x,0)=0,\quad\left.\frac{\partial\Theta^*}{\partial t}\right|_{t=0}=0
\end{cases}. \tag{4.14}$$

where $T_0^*(x)=T(x,15)$. Let

$$\Theta_1^*(x,t)=\Theta^*(x,t)-\Theta_2^*(x,t)=\Theta^*(x,t)-\left(1-\frac{x}{l}\right)g_2(t).$$

The problem regarding $\Theta_1^*(\mathrm{x,t})$ is thus

$$\begin{cases}
\dfrac{1}{\tau_0}\dfrac{\partial\Theta_1^*}{\partial t}+\dfrac{\partial^2\Theta_1^*}{\partial t^2}=A^2\dfrac{\partial^2\Theta_1^*}{\partial x^2}+B^2\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2\Theta_1^*}{\partial x^2}\right)+f^*(x,t)\\
\Theta_1^*(0,t)=\Theta_1^*(l,t)=0\\
\Theta_1^*(x,0)=\left(\dfrac{x}{l}-1\right)g_2(0)=\varphi^*(x)\\
\left.\dfrac{\partial\Theta_1^*}{\partial t}\right|_{t=0}=\left(\dfrac{x}{l}-1\right)g'_2(0)=\psi^*(x)
\end{cases} \tag{4.15}$$

where

$$f^*(x,t)=\left(\frac{x}{l}-1\right)\left[\frac{1}{\tau_0}g'_2(t)+g''_2(t)\right]+\frac{1}{\tau_q}\left\{\alpha\frac{d^2T_0^*}{dx^2}+\frac{\rho_b c_b\omega_b}{\rho c}\left[T_a-T_0^*(x)\right]+\frac{Q_m}{\rho c}\right\}\mathrm{e}^{\frac{\rho_b c_b\omega_b}{\rho c}t}$$

Here $g_2(t)$ is defined in (4.14). The solution of problem (4.15) has the same structure as (4.13) by replacing $\psi(x)$, $\varphi(x)$, and $f(x,t)$ with $\psi^*(x)$, $\varphi^*(x)$, and $f^*(x,t)$, respectively. $\Theta^*(x,t)$ and $T(x,t)$ can also be consequently obtained.

Figures 2 and 3 illustrate the temperature variation with time at the epidermis-dermis (ED) interface and dermis-fat (DF) interface, respectively. Figures 4 and 5 show the temperature profiles in the skin at the end of heating process ($t=15$ s) and the end of cooling process ($t=45$ s), respectively. The temperature profiles calculated from different bioheat models deviate significantly from each other. In particular, wave-front phenomenon is obvious for the thermal wave model. The comparison in Figs. 3–5 between the analytical solutions under two sets of parameters and the numerical solutions for multi-layer structure clearly shows that by carefully choosing values of skin thermophysical parameters, single-layer model is capable of predicting accurately those from multi-layer models. Note also that analytical results are much less time-consuming and more accurate than numerical ones (evident from the results of TWMBT).

Figure 6 shows the temperature variation with time at DF interface for different values of τ_q in thermal wave model. For a larger value of τ_q, a longer time is needed for the prediction of thermal wave model converges into that of the

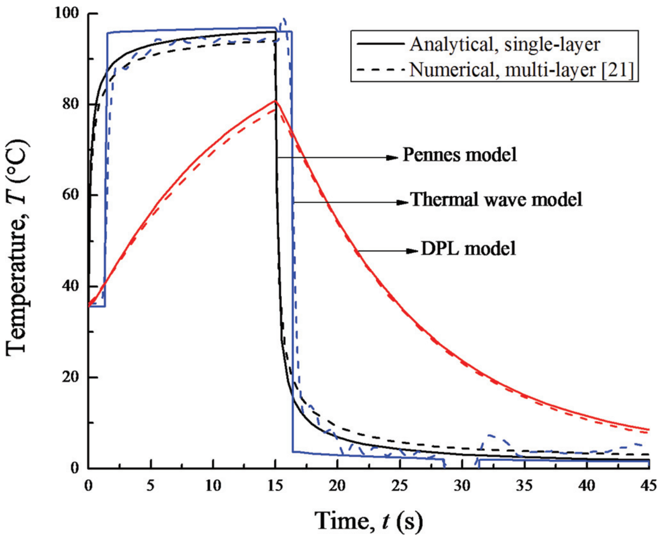

FIG. 2. (Color online) Temperature profiles at the ED interface.

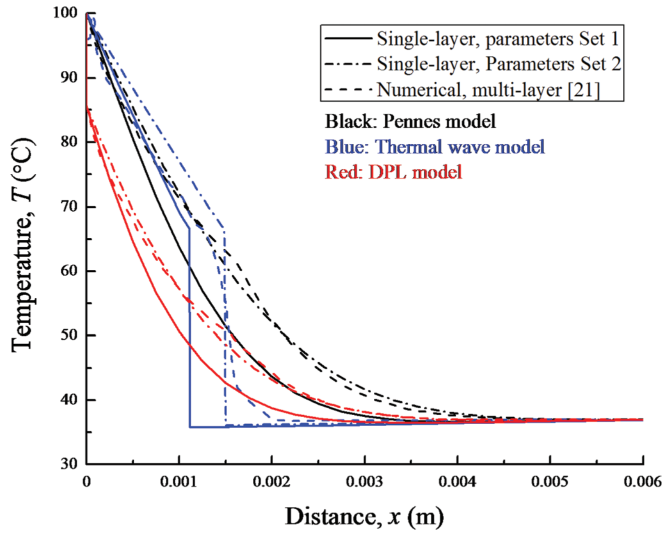

FIG. 4. (Color online) Temperature profiles in the skin at the end of heating process $t = 15$ s .

Pennes model. When $\tau_q = 1$ s, for example, the prediction of thermal wave model begins to agree with that of Pennes model after around 5 s; when $\tau_q = 3$ s, the two predictions become very close after around 10 s. When $\tau_q = 10$ s, however, large discrepancy still exists between the two predictions even after 30 s. The temperature variation with τ_T is illustrated in Fig. 7 at DF interface and with $\tau_q = 10$ s, showing that τ_T smoothes the τ_q- induced wave front and leads to a non-Fourier diffusionlike behavior of the skin temperature.

B. Magnetic hyperthermia

Magnetic hyperthermia therapy is very promising for treating some kinds of cancer, during which the temperature of cancerous cells is elevated above at least 42°C and maintains for approximately 30 mins by importing magnetic nanoparticles into the tumor tissue and applying an alternating magnetic field.[28,60,61] Highly selective heating of the cancerous tissue becomes achievable even for deep tumors in the body due to: (i) the possibility of selective concentration of magnetic nanoparticles in the tumor; (ii) high capability of magnetic particles to produce heat under the applied magnetic field; and (iii) the transparency of human tissues to magnetic fields.[60] To ensure that the effective treatment temperature ($> 42\,^\circ$C) localizes at the tumor region with little dissipation to the surrounding healthy tissue, a reliable prediction of the temperature profile during the magnetic hyperthermia therapy is of vital importance, according to which the magnetic field intensity and volume fraction of magnetic particles can be properly defined. In this section, the temperature profile within and around a spherical tumor is investigated as another example of the application of the solution structure theorems.

Consider a spherical tumor with radius of $r_0 = 3$ mm and located deeply in the body. Magnetic nanoparticles are either uniformly distributed in the tumor or superficially distributed near the tumor surface. The former distribution can be obtained by directly injecting magnetic nanofluids into the tumor at a low rate.[61,62] The latter, by the contrast, can be implemented by injecting the magnetic nanofluids through the artery

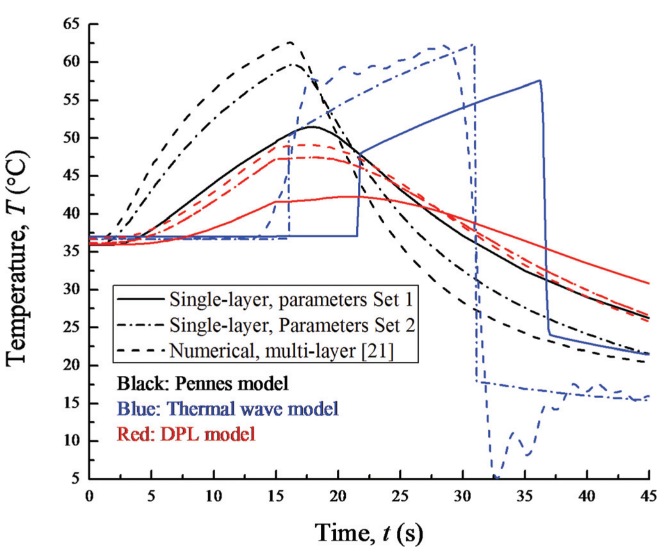

FIG. 3. (Color online) Temperature profiles at the DF interface.

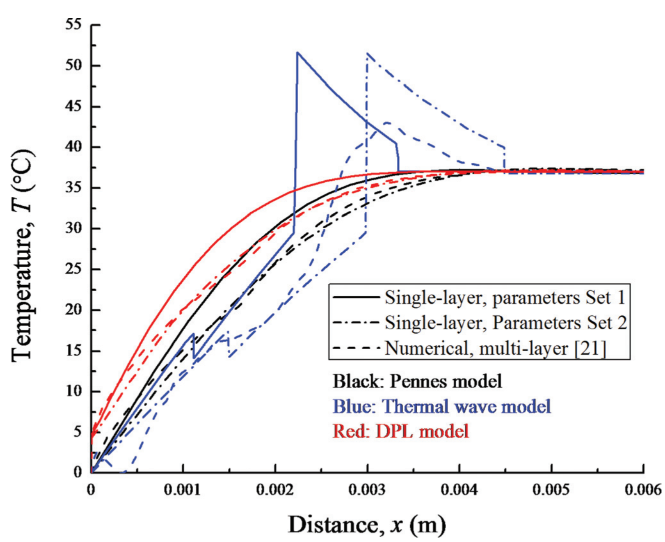

FIG. 5. (Color online) Temperature profiles in the skin at the end of cooling process ($t = 45$ s).

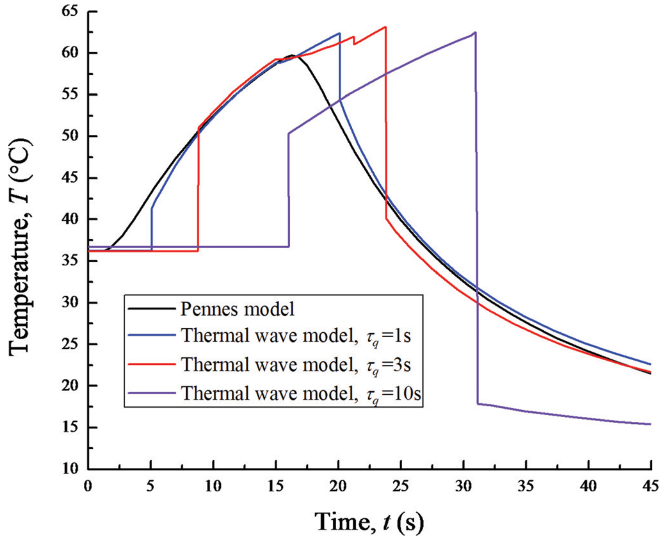

FIG. 6. (Color online) Temperature profiles at DF interface for different values of in thermal wave model.

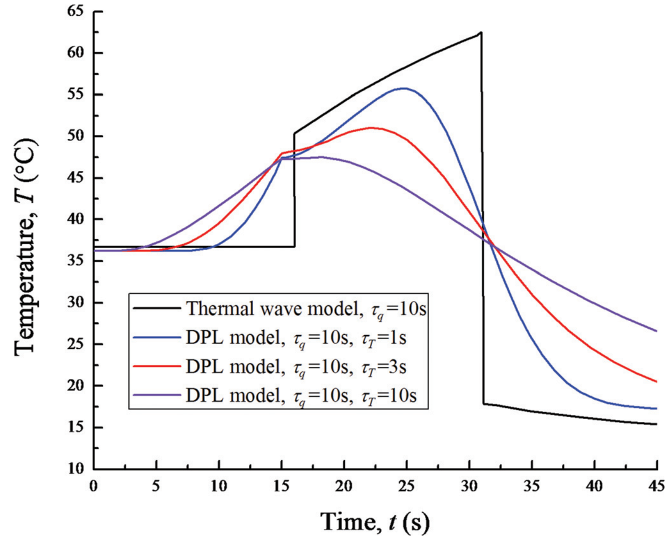

FIG. 7. (Color online) Temperature profiles at DF interface for different values of in the DPL model.

supplying to the tumor with specific binders attached to the surface of the nanoparticles.[61] By applying a magnetic field, the nanoparticles can generate heat within the tumor serving as an external supplied heat source. The governing equation, boundary conditions, and initial conditions are given by:

Pennes model:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) + \frac{\rho_b c_b \omega_b}{\rho c}[T_a - T(r,t)] + \frac{Q_m + Q_e}{\rho c} \tag{4.16}$$

Thermal wave model:

$$\left(\frac{1}{\tau_q} + \frac{\rho_b c_b \omega_b}{\rho c}\right)\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = \frac{\alpha}{\tau_q}\left(\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) + \frac{1}{\tau_q}\frac{\rho_b c_b \omega_b}{\rho c}[T_a - T(r,t)] + \frac{1}{\tau_q}\frac{Q_m + Q_e}{\rho c} \tag{4.17}$$

DPL model:

$$\left(\frac{1}{\tau_q} + \frac{\rho_b c_b \omega_b}{\rho c}\right)\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2} = \frac{\alpha}{\tau_q}\left(\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) + \alpha\frac{\tau_T}{\tau_q} \times \frac{\partial}{\partial t}\left(\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) + \frac{1}{\tau_q}\frac{\rho_b c_b \omega_b}{\rho c} \times [T_a - T(r,t)] + \frac{1}{\tau_q}\frac{Q_m + Q_e}{\rho c} \tag{4.18}$$

Boundary conditions:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0, \quad \left.\frac{\partial T}{\partial r}\right|_{r=R_0} = 0 \tag{4.19}$$

Initial conditions:

$$T(r,0) = T_0(r) : \begin{cases} \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right] + \frac{\rho_b c_b \omega_b}{\rho c}[T_a - T_0(r)] + \frac{Q_m}{\rho c} = 0, \\ \left.\frac{dT_0}{dr}\right|_{r=0} = 0, \quad \left.\frac{dT_0}{dr}\right|_{r=R_0} = 0 \end{cases} \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = 0. \tag{4.20}$$

Here r denotes the distance to the sphere center. R_0 is the radius of region for calculation which is set to be 10 times of the tumor's radius to satisfy the adiabatic boundary condition at $r = R_0$. The thermophysical properties of tumor and surrounding tissue are assumed to be the same and given in Table IV.[61] For the thermal wave model, $\tau_q = 16$s; for the DPL model, $\tau_q = 16$sand$\tau_T = 10$s. Again, the Pennes model and thermal wave model can be regarded as the two special cases of the DPL model at $\tau_q = \tau_T = 0$ and $\tau_q > \tau_T = 0$, so that we can focus our discussion on the solution of Eq. (4.18).

By solving the problem in (4.20), we can obtain the initial constant temperature $T_0(r) = T_a + Q_m/(\rho_b c_b \omega_b)$. To have a $T(r,t)$-independent source term in (4.18), introduce a function transformation:

$$\Theta(r,t) = [T(r,t) - T_0(r)]\mathrm{e}^{\frac{\rho_b c_b \omega_b}{\rho c} r}.$$

The problem regarding $\Theta(r,t)$ thus reads:

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial \Theta}{\partial t} + \dfrac{\partial^2 \Theta}{\partial t^2} = A^2 \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \Theta}{\partial r}\right) + B^2 \dfrac{\partial}{\partial t}\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial \Theta}{\partial r}\right)\right] + f(t) \\ \left.\dfrac{\partial \Theta}{\partial r}\right|_{r=0} = 0, \quad \Theta(R_0,t) = 0 \\ \Theta(r,0) = 0, \quad \left.\dfrac{\partial \Theta}{\partial t}\right|_{t=0} = 0, \end{cases} \tag{4.21}$$

where

$$\frac{1}{\tau_0} = \left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\right), \quad A^2 = \alpha\left(\frac{1}{\tau_q} - \frac{\rho_b c_b \omega_b}{\rho c}\frac{\tau_T}{\tau_q}\right),$$
$$B^2 = \alpha \frac{\tau_T}{\tau_q}, \quad f(r,t) = \frac{1}{\tau_q}\frac{Q_e(r)}{\rho c}\mathrm{e}^{\frac{\rho_b c_b \omega_b}{\rho c} t}.$$

According to the solution structure theorem, once we obtain the solution $W_\psi(r,t)$ of the problem:

$$\begin{cases} \dfrac{1}{\tau_0}\dfrac{\partial \Theta}{\partial t} + \dfrac{\partial^2 \Theta}{\partial t^2} = A^2 \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \Theta}{\partial r}\right) + B^2 \dfrac{\partial}{\partial t}\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial \Theta}{\partial r}\right)\right] \\ \left.\dfrac{\partial \Theta}{\partial r}\right|_{r=0} = 0, \quad \Theta(R_0,t) = 0 \\ \Theta(r,0) = 0, \quad \left.\dfrac{\partial \Theta}{\partial t}\right|_{t=0} = \psi(r). \end{cases} \tag{4.22}$$

The solution of the problem (4.21) can be readily written by $\int_0^t W_{f_\tau}(r, t-\tau)d\tau$.

By separating $\Theta(r,t)$ as $R(r)\Gamma(t)$ and substituting it into Eq. (4.22), we have

$$\frac{\Gamma''(t) + \frac{1}{\tau_0}\Gamma'(t)}{A^2\Gamma(t) + B^2\Gamma'(t)} = \frac{R''(r) + \frac{2}{r}R'(r)}{R(r)} = -\lambda,$$

where $-\lambda$ is the separation constant. The governing model for $R(r)$ is thus:

$$\begin{cases} R'' + \dfrac{2}{r}R' + \lambda R = 0 \\ \left.\dfrac{\partial R}{\partial r}\right|_{r=0} = 0, \quad R(R_0) = 0. \end{cases} \tag{4.23}$$

TABLE IV. Thermophysical properties of the tissue and blood.

Parameters	Values
Thermal conductivity, k (W/m K)	0.5
Density, ρ (kg/m^3)	1000
Specific heat, c (J/kg K)	3800
Blood density, ρ_b (kg/m^3)	1000
Blood specific heat, c_b (J/kg K)	3800
Metabolic heat generation, Q_m (W/m^3)	700
Arterial blood temperature, T_a (°C)	37
Blood perfusion rate, ω_b (s^{-1})	0.0005

In Sec. III A 4, we have shown that the solution of (4.23) can be written by a spherical Bessel function $j_0(\lambda_n^{1/2} r)$, where $\lambda_n = \left(\mu_n^{(1/2)}/r_0\right)^2$, $\mu_n^{(1/2)}$ are positive zero-points of $J_{1/2}(x)$, $n = 1, 2, 3, \ldots$.

For the present problem, since $\Delta = \left(1/\tau_0 + \lambda_n^2 B^2\right)^2 - 4\lambda_n^2 A^2 > 0$, the solution of $\Theta(r,t)$ can be written as

$$\Theta(r,t) = \sum_{n=1}^{\infty}(a_n \mathrm{e}^{r_1 t} + b_n \mathrm{e}^{r_2 t}) j_0\left(\mu_n^{(1/2)}/r_0 \cdot r\right),$$

where

$$r_{1,2} = -\frac{1}{2}\left(\frac{1}{\tau_0} + \lambda_n B^2\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau_0} + \lambda_n B^2\right)^2 - 4\lambda_n A^2}.$$

The initial conditions $\Theta(r,0) = 0$, $\frac{\partial \Theta}{\partial t}\big|_{t=0} = \psi(r)$ lead to

$$\begin{cases} a_n + b_n = 0 \\ a_n = \dfrac{1}{(r_1 - r_2)M_n}\displaystyle\int_0^{R_0} \psi(r) j_0\left(\mu_n^{(1/2)}/r_0 \cdot r\right) r^2 dr \\ \text{where } M_n = \displaystyle\int_0^{R_0} j_0^2\left(\mu_n^{(1/2)}/r_0 \cdot r\right) r^2 dr = \dfrac{\pi R_0^3}{4\mu_n^{(1/2)}} J_{3/2}^2\left(\mu_n^{(1/2)}\right). \end{cases}$$

Therefore, the solution of the problem (4.22) is

$$\mathrm{W}_\psi(r,t) = \sum_{n=1}^{\infty}\frac{1}{(r_1 - r_2)M_n}\int_0^{R_0} \psi(r) j_0\left(\mu_n^{(1/2)}/r_0 \cdot r\right) \times r^2 dr (\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t}) j_0\left(\mu_n^{(1/2)}/r_0 \cdot r\right).$$

Then the solution of the original problem (4.21) can be written as

$$\begin{aligned} \int_0^t W_{f_\tau}(r, t-\tau)d\tau &= \sum_{n=1}^{\infty}\frac{1}{(r_1 - r_2)M_n}\int_0^t\int_0^{R_0} f(\xi,\tau) \\ &\quad \times j_0\left(\mu_n^{(1/2)}/r_0 \cdot \xi\right)\xi^2 d\xi\left[\mathrm{e}^{r_1(t-\tau)} - \mathrm{e}^{r_2(t-\tau)}\right]d\tau \\ &= \frac{1}{\tau_q \rho c}\sum_{n=1}^{\infty}\frac{1}{(r_1 - r_2)M_n}\int_0^t \mathrm{e}^{\frac{\rho_b c_b \omega_b}{\rho c}\tau} \\ &\quad \times \left[\mathrm{e}^{r_1(t-\tau)} - \mathrm{e}^{r_2(t-\tau)}\right]d\tau \times \int_0^{R_0} Q_e(r) \\ &\quad \times j_0(\mu_n^{(1/2)}/r_0 \cdot \xi)\xi^2 d\xi j_0(\mu_n^{(1/2)}/r_0 \cdot r). \end{aligned} \tag{4.24}$$

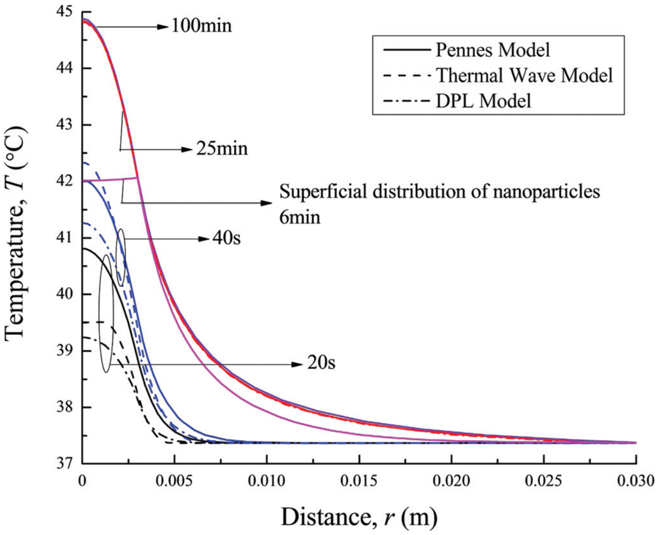

FIG. 8. (Color online) Temperature profiles under the uniform distribution of magnetic nanoparticles.

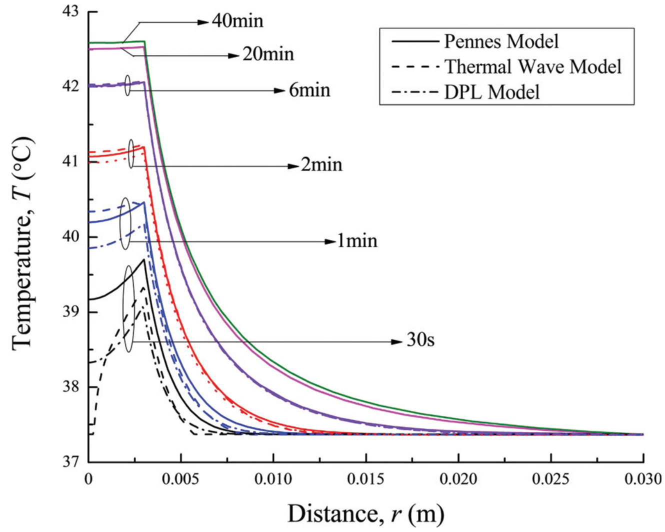

FIG. 9. (Color online) Temperature profiles under the superficial distribution of magnetic nanoparticles.

For the uniform distribution of magnetic particles,

$$Q_e(r) = \begin{cases} 9.5 \times 10^5 W/m^3, & r \le r_0 \\ 0, & r_0 < r \le R_0. \end{cases}$$

For the superficial distribution of magnetic particles,

$$\begin{cases} Q_e(r) = \dfrac{g_P}{4\pi r^2}\delta(r - r_0) \\ g_P = 0.12W, \end{cases}$$

where $\delta(x)$ represents the Dirac Delta function.

The temperature profiles as a function of r from three models are compared in Fig. 8 under the uniform distribution of magnetic nanoparticles. Obvious discrepancy appears at the early stage of hyperthermia treatment, e.g., and $t = 40$ s, due to the lagging effects of τ_q and τ_T. It is interesting to note that the temperature at the tumor center from thermal wave model is lower than that from Pennes model at $t = 20$ s but becomes higher at $t = 40$ s. When the time scale is much larger than that of τ_q and τ_T, temperature profiles from all the three models are almost the same. Note also that it may need a long time (around 25 min in this case) for the tumor surface temperature is over 42 °C from the instant when the tumor center temperature reaches 42 °C.

Figure 9 shows the temperature variation with r from three models under the superficial distribution of magnetic nanoparticles. At the early stage, large discrepancy exists among the three models. In particular, wave front is obvious for the thermal wave model. When the time scale is one order longer than that of τ_q and τ_T, temperature profiles from the three models become very close. The temperatures at different positions within the tumor can exceed 42 °C at almost the same time ($t = 360$ s in this case), which is advantageous than the uniform distribution of magnetic nanoparticles.

Besides the parameters examined here, blood perfusion rate ω_b could also have a significant effect on the temperature profiles within and surrounding the tumor. In real applications, the magnetic field intensity and amount of applied nanoparticles should be carefully adjusted to ensure effective-treatment temperature in the tumor and safe temperature in the healthy tissue, no matter which kind of nanoparticle distribution is applied.

V. CONCLUDING REMARKS

Macroscale bioheat transport models have been developed either by the mixture theory of continuum mechanics or by the porous-media theory. In the former, the global balance equations are scaled down; the required constitutive relations for heat flux are supplied directly at macroscale by the Fourier's law, the Cattaneo-Vernotte theory or the DPL relation. The thermal models developed in this approach contain, for example, Pennes model, Wulff model, Klinger model, Chen and Holmes model, thermal wave bioheat model, and DPL bioheat model. In the latter, both conservation and constitutive equations are introduced at the microscale. The method of volume averaging is then used to scale up the microscale equations and hence obtain the macroscale model. In order to form a closed system, the closure model must be provided for the unclosed terms that represent the microscale effect in macroscale field equations. Compared with the mixture theory of continuum mechanics, the porous-media approach is more powerful in offering connections between microscale and macroscale properties and accurately describing the rich blood-tissue interaction in biological tissues.

By using the porous-media approach, a general bioheat transport model is developed with the required closure provided. The model shows that both blood and tissue temperatures satisfy the DPL energy equations at macroscale. Due to the coupled conduction between blood and tissue, thermal waves and possible resonance may appear in bioheat transport. The blood-tissue interaction leads to very rich effects of the interfacial convective heat transfer, the blood velocity, the perfusion, and the metabolic reaction on blood and tissue macroscale temperature fields. Examples include: (i) the spreading of tissue metabolic effect into the blood DPL bioheat equation, (ii) the appearance of the convection term in the tissue DPL bioheat equation due to the blood velocity, and (iii) the appearance of sophisticated heat source terms in energy equations for blood and tissue temperatures.

DPL bioheat equations enjoy a very beautiful solution structure under linear boundary conditions: inter-expressible contributions of the initial temperature distribution, the source term and the initial rate of the change of temperature. Eleven solution structure theorems are developed in Cartesian, polar, cylindrical, and spherical coordinates for expressing solutions due to initial temperature distribution and source term by solutions due to the initial rate of temperature changes. They form a powerful tool for effectively resolving the DPL bioheat equations. This has been verified by the study of bioheat transport in skin tissue and during magnetic hyperthermia which has also revealed characteristics of different bioheat models and exemplified rich features of bioheat transport processes.

ACKNOWLEDGMENTS

The financial support from the Research Grants Council of Hong Kong (GRF718009) is gratefully acknowledged.

[1] *Animals and Temperature: Phenotypic and Evolutionary Adaptation*, edited by K. A. Johnston and A. F. Bennett, Society for Experimental Biology Seminar Series: 59 (Cambridge University Press, Cambridge, 1996).
[2] K. R. Diller, J. W. Valvano, and J. A. Pearce, in *The CRC Handbook of Thermal Engineering*, edited by F. Kreith (CRC Press, Boca Raton, 2000), pp. 4–114.
[3] L. Zhu, L. X. Xu, and N. Chencinski, IEEE Trans. Bio. Eng. **45**, 1163 (1998).
[4] A. A. Gage and J. Baust, Cryobiology **37**, 171 (1998).
[5] J. H. Zhou, J. K. Chen, and Y. W. Zhang, Computers in Biology and Medicine **39**, 286 (2009).
[6] Z. S. Deng and J. Liu, Computers in Biology and Medicine **34**, 495 (2004).
[7] H. H. Pennes, J. Appl. Phys. **1**, 93 (1948).
[8] W. Wulff, IEEE Trans. Bio. Eng. **21**, 494 (1974).
[9] H. G. Klinger, Bulletin of Mathematical Biology **36**, 403 (1974).
[10] H. G. Klinger, Bulletin of Mathematical Biology **38**, 183 (1976).
[11] M. M. Chen and K. R. Holmes, Annals of the New York Academy of Sciences **335**, 137 (1980).
[12] C. Cattaneo, Compte Rendus **247**, 431 (1958).
[13] P. Vernotte, Compte Rendus **246**, 3154 (1958).
[14] P. Vernotte, Compte Rendus **252**, 2190 (1961).
[15] D. Y. Tzou, in *Annual Review of Heat Transfer*, edited by C. L. Tien, Vol. **4**, p. 111 (1992).
[16] L. Q. Wang, X. S. Zhou, and X. H. Wei, *Heat Conduction: Mathematical Models and Analytical Solutions* (Springer-Verlag, Berlin Heidelberg, 2008).
[17] J. Liu, X. Chen, and L. X. Xu, IEEE Transactions on Biomedical Engineering **46**, 420 (1999).
[18] T. C. Shih, H. S. Kou, C. T. Liauh, and W. L. Lin, Med. Phys. **32**, 3029 (2005).
[19] M. M. Tung, M. Trujillo, J. A. L. Molina, M. J. Rivera, and E. J. Berjano, Mathematical and Computer Modelling **50**, 665 (2009).
[20] Z. S. Deng and J. Liu, Journal of Thermal Stresses **26**, 779 (2003).
[21] F. Xu, K. A. Seffen, and T. J. Lu, International Journal of Heat and Mass Transfer **51**, 2237 (2008).
[22] F. Xu, M. Lin, and T. J. Lu, Computers in Biology and Medicine **40**, 478 (2010).
[23] F. Xu, T. J. Lu, K. A. Seffen, and E. Y. K. Ng, Appl. Mech. Rev. **62**, 050801 (2009).
[24] W. Roetzel, N. Putra, and S. K. Das, International Journal of Thermal Sciences **42**, 541 (2003).
[25] K. Mitra, S. Kumar, A. Vedavarz, and M. K. Moallemi, Journal of Heat Transfer-Transactions of the Asme **117**, 568 (1995).
[26] D. Y. Tzou, *Macro- to Microscale Heat Transfer: The Lagging Behavior* (Taylor & Francis, Washington D. C., 1997).
[27] D. Y. Tzou, Journal of Heat Transfer-Transactions of the Asme **117**, 8 (1995).
[28] K. C. Liu and H. T. Chen, International Journal of Heat and Mass Transfer **52**, 1185 (2009).
[29] D. Y. Tzou, in *Advances in Transport Phenomenon*, edited by L. Q. Wang (Springer, New York, 2010), Vol. 2010.
[30] J. H. Zhou, Y. W. Zhang, and J. K. Chen, International Journal of Thermal Sciences **48**, 1477 (2009).
[31] S. Weinbaum and L. M. Jiji, Journal of Biomechanical Engineering-Transactions of the Asme **107**, 131 (1985).
[32] C. K. Charny and R. L. Levin, Journal of Biomechanical Engineering-Transactions of the Asme **111**, 263 (1989).
[33] J. C. Chato, ASME Journal of Biomechanical Engineering **102**, 110 (1980).
[34] H. W. Huang, Z. P. Chen, and R. B. Roemer, Journal of Biomechanical Engineering-Transactions of the Asme **118**, 120 (1996).
[35] K. H. Keller and L. Seiler, J. Appl. Phys. **30**, 779 (1971).
[36] L. M. Jiji, S. Weinbaum, and D. E. Lemons, Journal of Biomechanical Engineering-Transactions of the Asme **106**, 331 (1984).
[37] A. V. Kuznetsov and D. A. Nield, Transport in Porous Media **77**, 447 (2009).
[38] D. A. Nield and A. V. Kuznetsov, International Journal of Heat and Mass Transfer **51**, 5534 (2008).
[39] D. A. Nield and A. V. Kuznetsov, Journal of Heat Transfer-Transactions of the ASME **131**,101005 (2009).
[40] P. F. Scholander and J. Krog, J. Appl. Phys. **10**, 405 (1957).
[41] S. Weinbaum, L. M. Jiji, and D. E. Lemons, Journal of Biomechanical Engineering-Transactions of the Asme **106**, 321 (1984).
[42] Y. M. Xuan and W. Roetzel, Chemical Engineering & Technology **20**, 268 (1997).
[43] A. R. A. Khaled and K. Vafai, International Journal of Heat and Mass Transfer **46**, 4989 (2003).
[44] A. Nakayama and F. Kuwahara, International Journal of Heat and Mass Transfer **51**, 3190 (2008).
[45] E. R. Clark, Physiological Reviews **18**, 229 (1938).
[46] D. R. Johnson, in Human Biology Lecture Notes (Faculty of Biological Sciences, University of Leeds).
[47] L. Q. Wang and J. Fan, ASME Journal of Heat Transfer **133**, 011010 (2010).
[48] L. Q. Wang, M. T. Xu, and X. H. Wei, Advances in Chemical Engineering **34**, 175 (2008).
[49] M. Quintard and S. Whitaker, in *Handbook of Porous Media*, edited by K. Vafai (Taylor & Francis, Boca Raton, 2000).
[50] S. Whitaker, *The Method of Volume Averaging* (Kluwer Academic, 1999).
[51] M. Quintard and S. Whitaker, Advances in Heat Transfer **23**, 369 (1993).
[52] S. Whitaker, Transport in Porous Media **25**, 27 (1996).
[53] M. Quintard and S. Whitaker, in *Handbook of Heat Transfer in Porous Media*, edited by K. Vafai (Marcel Dekker, 2000), p. 1.
[54] M. Quintard, M. Kaviany, and S. Whitaker, Advances in Water Resources **20**, 77 (1997).
[55] J. Fan and L. Q. Wang, NANO **5**, 117 (2010).
[56] J. Fan and L. Q. Wang, J. Phys. D-Appl. Phys. **43**, 165501 (2010).
[57] M. T. Xu and L. Q. Wang, International Journal of Heat and Mass Transfer **45**, 1055 (2002).
[58] L. Wang and X. Wei, ASME Journal of Heat Transfer **131**, 033102 (2009).
[59] L. Q. Wang and M. Quintard, in *Advances in Transport Phenomena*, edited by L. Q. Wang (Springer, New York, 2009), Vol. 2009, p. 179.
[60] G. Bellizzi and O. M. Bucci, International Journal of Hyperthermia **26**, 389 (2010).
[61] M. A. Giordano, G. Gutierrez, and C. Rinaldi, International Journal of Hyperthermia **26**, 475 (2010).
[62] M. Salloum, R. H. Ma, D. Weeks, and L. Zhu, International Journal of Hyperthermia **24**, 337 (2008).